M353 H5 (S. Zhang) 2.6 3.1 3.2.

1. 2.6: 1,2,4

1. (2.6:1a) Show A is positive definite.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

- **ans:** If $x \neq 0$,

$$x^T Ax = x_1^2 + 3x_2^2 > 0$$

2. (2.6:1b) Show A is positive definite.

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 10 \end{pmatrix}$$

- **ans:** If $x \neq 0$,

$$\begin{aligned} x^T Ax &= x_1^2 + 6x_1x_2 + 10x_2^2 \\ &= (x_1 + 3x_2)^2 + x_2^2 > 0 \end{aligned}$$

3. (2.6:1c) Show A is positive definite.

$$A = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$$

- **ans:** If $x \neq 0$,

$$x^T Ax = x_1^2 + 2x_2^2 + 3x_3^2 > 0$$

4. (2.6:2a) Show A is not positive definite.

$$A = \begin{pmatrix} 1 & \\ & -3 \end{pmatrix}$$

- **ans:**

$$x^T Ax = x_1^2 - 3x_2^2;$$

$$x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x^T Ax = -3 < 0$$

5. (2.6:2b) Show A is not positive definite.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

- **ans:**

$$x^T Ax = x_1^2 + 4x_1x_2 + 2x_2^2 = (x_1 + x_2)^2 - 2x_2^2$$

$$x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x^T Ax = -2 < 0$$

6. (2.6:2c) Show A is not positive definite.

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$$

- **ans:**

$$x^T Ax = x_1^2 - 2x_1x_2 = (x_1 - x_2)^2 - x_2^2$$

$$x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x^T Ax = -1 < 0$$

7. (2.6:2d) Show A is not positive definite.

$$A = \begin{pmatrix} 1 & & \\ & -2 & \\ & & 3 \end{pmatrix}$$

- **ans:**

$$x^T Ax = x_1^2 - 2x_2^2 + x_3^2$$

$$x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad x^T Ax = -2 < 0$$

8. (2.6:4a) Do CG iteration with $x_0 = [0\ 0]$.

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \ b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- **ans:**

```
clc;  b=[0 1]'; x=0*b; A=[1 -1; -1 2];
  d=b, r=b,

  al=(r'*r)/(d'*A*d), rats(al)
  x = x + al * d
  r1 = r  -  al* A *d
  ba = (r1'*r1) /(r'*r)
  d = r1  + ba * d
  r=r1;

  al=(r'*r)/(d'*A*d), rats(al)
  x = x + al * d
  r1 = r  -  al* A *d
```

$$d_0 = r_0 = x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

first iteration

$$\alpha_1 = \frac{r_0^T r_0}{d_0^T A d_0} = \frac{1}{2}$$

$$x_1 = x_0 + \alpha_1 d_0 = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$$

$$r_1 = r_0 - \alpha_1 A d_0 = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

$$\beta_1 = r_0^T r_0/(r_1^T r_1) = \frac{1}{4}$$

$$d_1 = r_1 + \beta_1 d_0 = \begin{pmatrix} 1/2 \\ 1/4 \end{pmatrix}$$

2nd iteration (at most 2 for 2 by 2 systems)

$$\alpha_2 = \frac{r_1^T r_1}{d_1^T A d_1} = 2$$

$$x_2 = x_1 + \alpha_2 d_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$r_2 = r_1 - \alpha_2 A d_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The exact solution is

$$x = x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

9. (2.6:4b) Do CG iteration with $x_0 = [0\ 0]$.

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix},\ b = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

- **ans:**

```
clc;  b=[-3 3]'; x=0*b; A=[4 1; 1 4];
  d=b, r=b,

  al=(r'*r)/(d'*A*d), rats(al)
  x = x + al * d
  r1 = r  -  al* A *d
  ba = (r1'*r1) /(r'*r)
  d = r1  + ba * d
  r=r1;

  al=(r'*r)/(d'*A*d), rats(al)
  x = x + al * d
  r1 = r  -  al* A *d
```

$$d_0 = r_0 = x_0 = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

first iteration

$$\alpha_1 = \frac{r_0^T r_0}{d_0^T A d_0} = \frac{1}{3}$$

$$x_1 = x_0 + \alpha_1 d_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$r_1 = r_0 - \alpha_1 A d_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

At most 2 steps for 2 by 2 systems. But we are lucky to get the exact solution in one step.

The exact solution is

$$x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

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1. 3.1: a1, a2, 1c, 2c, 3, 5, 8

1. (3.1:a1) Find the $P_2(x)$ interpolation by
(1) solving equations for unknown coefficients,
(2) Lagrange nodal basis,
(3) Newton's divided differences.

x_i	-1	0	1
y_i	0	-2	0

- **ans:**

$$P = 2x^2 - 2$$

Method(1) We look for a degree 2 polynomial to fit 3 data points.

$$P_3(x) = a + bx + cx^2$$

For example, when $x_0 = -1$, $y_0 = 0$ we get one equation:

$$0 = a - b + c$$

Then for the unknown coefficients (vector $X = (a, b, ...)$), we get the following equations obtained by evaluating the polynomial at the given points:

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$$

Solve the system of equations:

$$X = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$$

$$P_3(x) = -2 + 2x^2$$

Method(2) Using Lagrange basis

$$\begin{aligned} P_2(x) =& y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \\ &+ y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \\ &+ y_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \\ =& 0 - 2\frac{(x + 1)(x - 1)}{(1)(-1)} + 0 \\ =& 2(x^2 - 1) \\ =& 2x^2 - 2 \end{aligned}$$

We must get the same answer. Also we can check the function by evaluate it at the 4 given point!

Method (3) Newton's divided differences:

$$
\begin{aligned}
f[x_k] &= f(x_k) \\
f[x_k, x_{k+1}] &= \frac{f[x_{k+1}] - f[x_k]}{x_{k+1} - x_k} \\
f[x_k, x_{k+1}, x_{k+2}] &= \frac{f[x_{k+1}, x_{k+2}] - f[x_k, x_{k+1}]}{x_{k+2} - x_k}
\end{aligned}
$$

$$
P = \sum_{i=0}^{n} f[x_0 ... x_i](x - x_0)...(x - x_{i-1})
$$

We can use the above formulas directly. But it is much easier to use the following table.

$$
\begin{pmatrix}
-1 & \boxed{0} & & \\
 & & \boxed{-2} & \\
0 & -2 & & \boxed{2} \\
 & & 2 & \\
1 & 0 & &
\end{pmatrix}
$$

$$
\begin{aligned}
P_3(x) &= 0 - 2(x+1) + 2(x+1)(x) \\
&= -2x - 2 + 2x^2 + 2x \\
&= 2x^2 - 2
\end{aligned}
$$

2. (3.1:a2) Find the $P_3(x)$ interpolation by
(1) solving equations for unknown coefficients,
(2) Lagrange nodal basis,
(3) Newton's divided differences.

x_i	-1	0	1	2
y_i	0	-2	0	0

- **ans:** Method(1) We look for a degree 3 polynomial to fit 4 data points.

$$
P_3(x) = a + bx + cx^2 + dx^3
$$

For example, when $x_0 = -1$, $y_0 = 0$ we get one equation:

$$
0 = a - b + c - d
$$

Then for the unknown coefficients (vector $X = (a, b, ...)$), we get the following equations obtained by evaluating the polynomial at the given points:

$$
\begin{pmatrix}
1 & -1 & 1 & -1 \\
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
1 & 2 & 4 & 8
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}
=
\begin{pmatrix} 0 \\ -2 \\ 0 \\ 0. \end{pmatrix}
$$

Solve the system of equations:

$$
X = \begin{pmatrix} -2 \\ 1 \\ 2 \\ -1 \end{pmatrix}
$$

$$
P_3(x) = -2 + x + 2x^2 - x^3
$$

Method(2) Using Lagrange basis

$$
\begin{aligned}
p_3(x) =& y_0 \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \\
&+ y_1 \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\
&+ y_2 \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \\
&+ y_3 \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \\
=& 0 - 2\frac{(x+1)(x-1)(x-2)}{(1)(-1)(-2)} + 0 + 0 \\
=& -(x^2 - 1)(x - 2) \\
=& -x^3 + 2x^2 + x - 2
\end{aligned}
$$

We must get the same answer. Also we can check the function by evaluate it at the 4 given point!

Method (3) Newton's divided differences:

$$
\begin{aligned}
f[x_k] &= f(x_k) \\
f[x_k, x_{k+1}] &= \frac{f[x_{k+1}] - f[x_k]}{x_{k+1} - x_k} \\
f[x_k, x_{k+1}, x_{k+2}] &= \frac{f[x_{k+1}, x_{k+2}] - f[x_k, x_{k+1}]}{x_{k+2} - x_k}
\end{aligned}
$$

$$
\begin{aligned}
& f[x_k, x_{k+1}, x_{k+2}, x_{k+3}]] \\
=& \frac{f[x_{k+1}, x_{k+2}, x_{k+3}] - f[x_k, x_{k+1}, x_{k+2}]}{x_{k+3} - x_k}
\end{aligned}
$$

$$
P = \sum_{i=0}^{n} f[x_0 ... x_i](x - x_0)...(x - x_{i-1})
$$

We can use the above formulas directly. But it is much easier to use the following table.

$$
\begin{pmatrix}
-1 & \boxed{0} & & & \\
 & & \boxed{-2} & & \\
0 & -2 & & \boxed{2} & \\
 & & 2 & & \boxed{-1} \\
1 & 0 & & -1 & \\
 & & 0 & & \\
2 & 0 & & &
\end{pmatrix}
$$

$$
\begin{aligned}
P_3 &= 0 - 2(x+1) + 2(x+1)(x) - (x+1)x(x-1) \\
&= -2x - 2 + 2x^2 + 2x - x^3 + x \\
&= -x^3 + 2x^2 + x - 2
\end{aligned}
$$

3. (3.1:1b) Find $P_3(x)$ by (1) solving equations, (2) Lagrange nodal basis,

x_i	-1	2	3	5
y_i	0	1	1	2

- **ans:** (1) solving equations,

$$P_3(x) = a + bx + cx^2 + dx^3$$

matching the given points:

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 5 & 5^2 & 5^3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0.4167 & 1.6667 & -1.2500 & 0.1667 \\ -0.4306 & 0.7778 & -0.3750 & 0.0278 \\ 0.1389 & -0.7778 & 0.7500 & -0.1111 \\ -0.0139 & 0.1111 & -0.1250 & 0.0278 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.7500 \\ 0.4583 \\ -0.2500 \\ 0.0417 \end{pmatrix}$$

So

$$\begin{aligned} P_3(x) =& 0.75 + 0.4583x \\ & - 0.25x^2 + 0.2500 + 0.0417x^3 \end{aligned}$$

(2) Lagrange nodal basis,

$$\begin{aligned} p_3(x) =& y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \\ &+ y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\ &+ y_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \\ &+ y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \\ =& \frac{(x+1)(x-3)(x-5)}{(2+1)(2-3)(2-5)} \\ &+ \frac{(x+1)(x-2)(x-5)}{(3+1)(3-2)(3-5)} \\ &+ 2\frac{(x+1)(x-2)(x-3)}{(5+1)(5-2)(5-3)} \\ =& 0.75 + 0.4583x \\ & - 0.25x^2 + 0.2500 + 0.0417x^3 \end{aligned}$$

4. (3.1:5) Interpolation the function by a degree 3 polynomial $P(x)$: at points

x	-2	0	1	3
y	8	4	2	-2

For P_d, $d = 4$, how many polynomials interpolate the function at the 4 points?

- **ans:** Use the method of Newton's divided differences.

$$\begin{aligned} f[x_k] &= f(x_k) \\ f[x_k, x_{k+1}] &= \frac{f[x_{k+1}] - f[x_k]}{x_{k+1} - x_k} \\ f[x_k, x_{k+1}, x_{k+2}] &= \frac{f[x_{k+1}, x_{k+2}] - f[x_k, x_{k+1}]}{x_{k+2} - x_k} \end{aligned}$$

$$\begin{aligned} & f[x_k, x_{k+1}, x_{k+2}, x_{k+3}]] \\ =& \frac{f[x_{k+1}, x_{k+2}, x_{k+3}] - f[x_k, x_{k+1}, x_{k+2}]}{x_{k+3} - x_k} \end{aligned}$$

$$P = \sum_{i=0}^{n} f[x_0 ... x_i](x - x_0)...(x - x_{i-1})$$

We make a table:

$$\begin{pmatrix} -2 & 8 & & & \\ & & -2 & & \\ 0 & 4 & & 0 & \\ & & -2 & & 0 \\ 1 & 3 & & 0 & \\ & & -2 & & \\ 3 & -2 & & & \end{pmatrix}$$

$$\begin{aligned} P(x) &= 8 - 2(x+2) + 0 + 0 \\ &= 4 - 2x \end{aligned}$$

$d = 4$, infinitely many polynomials, one parameter

$$\begin{aligned} & P_d(x) = P(x) + Q(x)(x+2)(x)(x-1)(x-3) \\ & Q(x) = A \end{aligned}$$

5. (3.1:8) Find $P_9(0)$ interpolating $f(x)$ at the given points, if $f(1) = 112$, $f(10) = 2$ and $f(i) = 0$, $i = 2, 3, 4, 5, 6, 7, 8, 9$.

- **ans:** 2 Methods.

(1) using Lagrange basis:

$$\begin{aligned} P_9(x) =& f(1)\frac{(x-2)(x-3)...(x-9)(x-10)}{(1-2)(1-3)...(1-9)(1-10)} + 0 + ... \\ &+ f(10)\frac{(x-1)(x-2)...(x-9)}{(10-1)(1-2)...(10-9)} \\ &= ... \end{aligned}$$

But the second method is better:

(2) Solving equations:

$$\begin{aligned} P_9(x) &= (A + Bx)(x-2)(x-3)(x-4)(x-5) \\ & \quad (x-6)(x-7)(x-8)(x-9) \\ 112 &= f(1) = (A + B)(8!) \\ 2 &= f(10) = (A + 10B)(8!) \end{aligned}$$

Equation 2 minus equation 1:

$$-110 = 9B(8!),\ B = -\frac{110}{9!}$$

Equation 2 minus 10 times equation 1:

$$2 - 1120 = -9A(8!),\ A = -\frac{1118}{9!}$$

$$P_9(0) = A(-9!) = 1118$$

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1. 3.2: 2, 4
2. (3.2:2) Let $f(x) = \ln x$. Let $x_i = 1, 2, 4$
   (a) Interpolate f at the three points by a P_2.
   (b) Find $f(3)$, $P_2(3)$, and the error .
   (c) Find the bound on error $e(x) = f(x) - P_2(x)$ by the Lagrange theory.
   (d) Compare error bound with actual error $e(3)$

   - **ans:**

   (a) We look for a degree 3 polynomial to fit 3 data points.

   $$P_2(x) = A + Bx + Cx^2$$

   Then for the unknown coefficients (vector $X = (A, B, ...)$), we get the following equations obtained by evaluating the polynomial at the given points:

   $$\begin{pmatrix} 1. & 1 & 1 \\ 1. & 2 & 2^2. \\ 1. & 4 & 4^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ \ln 2 \\ \ln 4 \end{pmatrix}$$

   Note that the rhs are values $f(x_i)$

   $$X = \begin{pmatrix} -0.9242 \\ 1.0397 \\ -0.1155 \end{pmatrix}$$

   $$P_2(x) = -0.9242 + 1.0397x - 0.1155x^2$$

   (b)

   $$e(3) = f(3) - P_2(3) = -0.0568$$

   (c)

   $$\begin{aligned} e(x) &= \frac{f'''(z)}{3!}(x - x_0)(x - x_1)(x - x_2) \\ f'''(z) &= 2z^{-3} \\ e(x) &= \frac{2z^{-3}}{6}(x - 1)(x - 2)(x - 4) \\ |e(3)| &= |\frac{2z^{-3}}{6}(-2)| \\ &\leq \frac{2(1^{-3})}{6}(2) = \frac{2}{3} = 0.66 \\ |e(3)| &= 0.0568 \text{ yes, smaller} \end{aligned}$$

3. (3.2:4) Let $f(x) = 1/(x + 5)$. Let $x_i = 0, 2, 4, 6, 8, 10$. Find the bound on errors $e(1)$ and $e(5)$, $e(x) = f(x) - P_5(x)$, by the Lagrange theory.

   - **ans:**

   $$\begin{aligned} e(x) &= \frac{f^{(6)}(z)}{6!}\prod_{i=0}^{5}(x - x_i) \\ f'(z) &= -(x + 5)^{-2} \\ f^{(6)}(z) &= 5!(x + 5)^{-6} \\ |f^{(6)}(z)| &\leq 5!(0 + 5)^{-6} \\ |e(1)| &\leq \frac{5!}{5^6 \cdot 6!}|1(1 - 2)(1 - 4)(1 - 6)(1 - 8)(1 - 10)| \\ &= 0.0504 \\ |e(5)| &\leq \frac{5!}{5^6 \cdot 6!}|5(5 - 2)(5 - 4)(5 - 6)(5 - 8)(5 - 10)| \\ &= 0.012 \end{aligned}$$
