M353 Hw 4 (S. Zhang) 2.5 .

1. (2.5:a1) Find x_2 by (a) Jacobi and by (b) Gauss–Seidel iterations, find (c) $\|R_j\|_\infty$, (d) $\|R_{gs}\|_\infty$, and (e) verify the error bounds $\|x - x_i\|_\infty \le \|R_j\|_\infty^i \|x - x_0\|_\infty$.

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

- **ans:**

(a) Jacobi:

$$\begin{aligned} x_1 &= D^{-1}(b - (L_t + U_t)x_0) \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1/2 & 0 \end{pmatrix} x_0 \\ &= \begin{pmatrix} 0 \\ 1.5 \end{pmatrix} \\ x_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1/2 & 0 \end{pmatrix} x_1 \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

(b) Gauss-Seidel:

$$\begin{aligned} x_1 &= (L_t + D)^{-1}(b - U_t x_0) \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} x_0 \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ x_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} x_1 \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

(c) Jacobi rate:

$$\begin{aligned} \|R_j\|_\infty &= \|D^{-1}(L_t + U_t)\|_\infty \\ &= \| \begin{pmatrix} 0 & 0 \\ -1/2 & 0 \end{pmatrix} \|_\infty \\ &= \max\{1/2, 0\} = 1/2 \end{aligned}$$

(d) Gauss-Seidel rate:

$$\begin{aligned} \|R_{gs}\|_\infty &= \|(L_t + D)^{-1} U_t\|_\infty \\ &= \max\{0, 0\} = 0 \end{aligned}$$

(e) Verify Jacobi error reduction bound:

Exact solution

$$x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Errors

$$\|x - x_0\|_\infty = \| \begin{pmatrix} -1 \\ 1 \end{pmatrix} \infty = 1$$

$$\|x - x_1\|_\infty = \| \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} \infty = 1/2$$

$$\|x - x_2\|_\infty = \| \begin{pmatrix} 0 \\ 0 \end{pmatrix} \infty = 0$$

Is

$$\|x - x_i\|_\infty \le \|R_j\|_\infty^i \|x - x_0\|_\infty ?$$

$i = 1$:

$$\frac{1}{2} \le \frac{1}{2}(1)? \text{ Yes!}$$

$i = 2$:

$$0 \le (\frac{1}{2})^2(1)? \text{ Yes!}$$

2. (2.5:a2) Let

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, x_0 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Find x_2 if Jacobi iteration is used.

(b) Find x_2 if Gauss–Seidel iteration is used.

(c) Show A is strictly diagonally dominated.

(d) Find the error reduction factor for the Jacobin iteration, $\|R_j\|_\infty$.

(e) Find the error reduction factor for the Gauss–Seidel iteration, $\|R_{gs}\|_\infty$.

- **ans:**

$$A = L_t + D + U_t$$

Jacobi:

$$\begin{aligned} x_1 &= D^{-1}b - D^{-1}(L_t + U_t)x_0 \\ &= \begin{pmatrix} 2 & & \\ & 3 & \\ & & 2 \end{pmatrix}^{-1} \left(\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \right. \\ &\quad \left. - \begin{pmatrix} & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & -1/3 \\ 0 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 \\ 1/3 \\ 3/2 \end{pmatrix} \\ x_2 &= D^{-1}b - D^{-1}(L_t + U_t)x_1 \\ &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & -1/3 \\ 0 & -1/2 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/3 \\ 3/2 \end{pmatrix} \\ &= \begin{pmatrix} 5/6 \\ 1/3 \\ 7/6 \end{pmatrix} \end{aligned}$$

Gauss-Seidel:

$$
\begin{aligned}
x_1 &= (D+L_t)^{-1}b - (D+L_t)^{-1}U_t x_0 \\
&= \begin{pmatrix} 2 & & \\ 1 & 3 & \\ 0 & -1 & 2 \end{pmatrix}^{-1} \Bigg(\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \\
&- \begin{pmatrix} & 1 & 0 \\ & 0 & -1 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Bigg) \\
&= \begin{pmatrix} 1 \\ -1/3 \\ 5/6 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 0 & 3 & 0 \\ 0 & -1 & -2 \\ 0 & -1/2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1/2 \\ 1/6 \\ 13/12 \end{pmatrix} \\
x_2 &= (D+L_t)^{-1}b - (D+L_t)^{-1}U_t x_1 \\
&= \begin{pmatrix} 1 \\ -1/3 \\ 5/6 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 0 & 3 & 0 \\ 0 & -1 & -2 \\ 0 & -1/2 & -1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/6 \\ 13/12 \end{pmatrix} \\
&= \begin{pmatrix} 33/36 \\ 2/36 \\ 37/36 \end{pmatrix}
\end{aligned}
$$

Strictly Diagonally Dominated:

$$
\begin{aligned}
2 &> 1 + 0 \\
3 &> 1 + |-1| \\
2 &> 0 + |-1|
\end{aligned}
$$

So Jacobi and Gauss-Seidel iterations converge.

Jacobi:

$$
\begin{aligned}
\|R_j\|_\infty &= \|D^{-1}(L_t + U_t)\|_\infty \\
&= \| \begin{pmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & -1/3 \\ 0 & -1/2 & 0 \end{pmatrix} \|_\infty \\
&= \max\{1/2, 1/3 + 1/3, 1/2\} = 2/3
\end{aligned}
$$

Gauss-Seidel:

$$
\begin{aligned}
\|R_{gs}\|_\infty &= \|(D+L_t)^{-1}U_t\|_\infty \\
&= \| \frac{1}{6} \begin{pmatrix} 0 & 3 & 0 \\ 0 & -1 & -2 \\ 0 & -1/2 & -1 \end{pmatrix} \|_\infty \\
&= 1/2
\end{aligned}
$$

Checking. Exact solution:

$$
x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}
$$

Error reduction for Jacobi:

$$
\begin{aligned}
\|e_0\|_\infty &= \| \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \|_\infty = 1 \\
\|e_1\|_\infty &= \| \begin{pmatrix} 1/2 \\ -1/3 \\ -1/2 \end{pmatrix} \|_\infty = 1/2 \\
\|e_2\|_\infty &= \| \begin{pmatrix} 1/6 \\ -1/3 \\ -1/6 \end{pmatrix} \|_\infty = 1/3
\end{aligned}
$$

$$
\begin{aligned}
\|e_1\|_\infty &\le \|R_j\|_\infty \|e_0\|_\infty ? \\
1/2 &\le (2/3)(1), \text{ yes} \\
\|e_2\|_\infty &\le \|R_j\|_\infty \|e_1\|_\infty ? \\
1/3 &\le (2/3)(1/2), \text{ yes}
\end{aligned}
$$

Error reduction for GS:

$$
\begin{aligned}
\|e_0\|_\infty &= \| \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \|_\infty = 1 \\
\|e_1\|_\infty &= \| \begin{pmatrix} 1/2 \\ -1/6 \\ -1/12 \end{pmatrix} \|_\infty = 1/2 \\
\|e_2\|_\infty &= \| \begin{pmatrix} 1/12 \\ -1/18 \\ -1/36 \end{pmatrix} \|_\infty = 1/18
\end{aligned}
$$

$$
\begin{aligned}
\|e_1\|_\infty &\le \|R_{gs}\|_\infty \|e_0\|_\infty ? \\
1/2 &\le (1/2)(1), \text{ yes} \\
\|e_2\|_\infty &\le \|R_{gs}\|_\infty \|e_1\|_\infty ? \\
1/18 &\le (1/2)(1/2), \text{ yes}
\end{aligned}
$$