M353 Hw 3 (S. Zhang) 2.2-4.

1. 2.2: 2,4,5,a1,a2

2. (2.2:2c) Find and check the LU factorization:

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 3 & 4 & 4 \\ 0 & 2 & 1 & -1 \end{pmatrix}$$

- **ans:** All row operations must be exactly as shown here. The boxed numbers are for matrix L, as if a computer re-uses the storage, where the elements are eliminated.

$$\begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 3 & 4 & 4 \\ 0 & 2 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{\substack{(0)r_1+r_2 \\ (-1)r_1+r_3 \\ (0)r_1+r_4}} \begin{pmatrix} 1 & -1 & 1 & 2 \\ \boxed{0} & 2 & 1 & 0 \\ \boxed{-1} & 4 & 3 & 2 \\ \boxed{0} & 2 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{\substack{(-2)r_2+r_3 \\ (-1)r_2+r_4}} \begin{pmatrix} 1 & -1 & 1 & 2 \\ \boxed{0} & 2 & 1 & 0 \\ \boxed{-1} & \boxed{2} & 1 & 2 \\ \boxed{0} & \boxed{1} & 0 & -1 \end{pmatrix}$$

$$\xrightarrow{(0)r_3+r_4} \begin{pmatrix} 1 & -1 & 1 & 2 \\ \boxed{0} & 2 & 1 & 0 \\ \boxed{-1} & \boxed{2} & 1 & 2 \\ \boxed{0} & \boxed{1} & \boxed{0} & -1 \end{pmatrix}$$

So

$$L = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 1 & 2 & 1 & \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 & 1 & 2 \\ & 2 & 1 & 0 \\ & & 1 & 2 \\ & & & -1 \end{pmatrix}$$

Check: $LU = A$:

$$\begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 1 & 2 & 1 & \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & 2 \\ & 2 & 1 & 0 \\ & & 1 & 2 \\ & & & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 3 & 4 & 4 \\ 0 & 2 & 1 & -1 \end{pmatrix}$$

3. (2.2:4a) Find the LU factorization, then use it with 2-step back substitution to solve the equation $Ax = b$ where

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

- **ans:** All row operations must be exactly as shown here. The boxed numbers are for matrix L, as if a computer re-uses the storage, where the elements are eliminated.

$$\begin{pmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{pmatrix}$$

$$\xrightarrow{\substack{(-2)r_1+r_2 \\ (-1)r_1+r_3}} \begin{pmatrix} 3 & 1 & 2 \\ \boxed{2} & 1 & 0 \\ \boxed{1} & 0 & 3 \end{pmatrix}$$

$$\xrightarrow{(0)r_2+r_3} \begin{pmatrix} 3 & 1 & 2 \\ \boxed{2} & 1 & 0 \\ \boxed{1} & \boxed{0} & 3 \end{pmatrix}$$

So

$$L = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ 1 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 1 & 2 \\ & 1 & 0 \\ & & 3 \end{pmatrix}$$

Solve by forward substitution, $Ly = b$:

$$\begin{pmatrix} 1 & & \\ 2 & 1 & \\ 1 & 0 & 1 \end{pmatrix} y = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

$$y = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

Solve by backward substitution, $Ux = y$:

$$\begin{pmatrix} 3 & 1 & 2 \\ & 1 & 0 \\ & & 3 \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

$$x = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

4. (2.2:a1) Let

$$(A|b) = \left(\begin{array}{ccccc} 2 & 1 & 1 & | & 3 \\ 4 & -3 & 4 & | & 1 \\ -6 & 1 & 2 & | & -5 \end{array}\right).$$

(a) Use Gaussian elimination without pivoting to solve $Ax = b$.

(b) Find $A = LU$ where L is the unit lower triangular matrix.

(c) Using LU decomposition of A to solve $Ax = b$.

- **ans:**

(a)

$$(A|b) \to (U|c)$$

$(-2)r_1 + r_2 \to r_2$, $(3)r_1 + r_3 \to r_3$:

$$\left(\begin{array}{ccccc} 2 & 1 & 1 & | & 3 \\ & -5 & 2 & | & -5 \\ & 4 & 5 & | & 4 \end{array}\right)$$

$(4/5)r_2 + r_3 \to r_3$:

$$\left(\begin{array}{ccccc} 2 & 1 & 1 & | & 3 \\ & -5 & 2 & | & -5 \\ & & 6.6 & | & 0 \end{array}\right)$$

Backward substitution, starting from equation 3.

$$x = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix}$$

(b) LU decomposition:
Repeating (a) (so that we can skip (a), otherwise we do not need the carry b in the row operations for finding LU), save row operations in the storage:

$$\left(\begin{array}{ccccc} 2 & 1 & 1 & | & 3 \\ \boxed{2} & -5 & 2 & | & -5 \\ -6 & 1 & 2 & | & -5 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc} 2 & 1 & 1 & | & 3 \\ \boxed{2} & -5 & 2 & | & -5 \\ \boxed{-3} & 4 & 5 & | & 4 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc} 2 & 1 & 1 & | & 3 \\ \boxed{2} & -5 & 2 & | & -5 \\ \boxed{-3} & \boxed{-.8} & 6.6 & | & 0 \end{array}\right)$$

$$x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{(bottom up)}$$

So, from the boxed numbers we get:

$$A = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ -3 & -.8 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ & -5 & 2 \\ & & 6.6 \end{pmatrix} = LU$$

(c)

$$Ly = b \Rightarrow y = \begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix} \quad \text{(top down)}$$

$$Ux = y \Rightarrow x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{(bottom up)}$$

5. (2.2:a1) Solve the following system $(A|b)$ by

(a) GE without pivoting, (no row switching, no row multiplication)

(b) finding $A = LU$ where L is the unit lower triangular matrix, and using LU decomposition of A to solve $Ax = b$ again.

$$\left(\begin{array}{ccccc} 1 & -3 & 3 & | & 4 \\ -2 & 0 & 1 & | & -1 \\ 3 & -1 & -1 & | & 2 \end{array}\right)$$

- **ans:**

(a) $2r_1 + r_2$, $-3r_1 + r_3$

$$\left(\begin{array}{ccccc} 1 & -3 & 3 & | & 4 \\ 0 & -6 & 7 & | & 7 \\ 0 & 8 & -10 & | & -10 \end{array}\right)$$

$(4/3)r_2 + r_3$.

$$\left(\begin{array}{ccccc} 1 & -3 & 3 & | & 4 \\ 0 & -6 & 7 & | & 7 \\ 0 & 0 & -0.66667 & | & -0.66667 \end{array}\right)$$

$$x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

(b) recording the "negative" operations above.
$2r_1 + r_2$, $-3r_1 + r_3$ then $(4/3)r_2 + r_3$.

$$L = \begin{pmatrix} 1 & & \\ -2 & 1 & \\ 3 & -4/3 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & -3 & 3 \\ 0 & -6 & 7 \\ 0 & 0 & -0.66667 \end{pmatrix}$$

$$Ly = b, \; y = \begin{pmatrix} 4 \\ 7 \\ -2/3 \end{pmatrix}$$

$$Ux = y, \; x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

6. (2.2:a2) Solve the following system $(A|b)$ by

(a) GE without pivoting, (no row switching, no row multiplication)

(b) finding $A = LU$ where L is the unit lower triangular matrix,

(c) using LU decomposition of A to solve $Ax = b$ again.

$$\left(\begin{array}{cccccc} 1 & 1 & 0 & -1 & | & 0 \\ -2 & 0 & 1 & 4 & | & -1 \\ 1 & 1 & -1 & 2 & | & -1 \\ 3 & 5 & 1 & -3 & | & -1 \end{array}\right)$$

- **ans:**

$$\begin{pmatrix} 1 & 1 & 0 & -1 & | & 0 \\ -2 & 0 & 1 & 4 & | & -1 \\ 1 & 1 & -1 & 2 & | & -1 \\ 3 & 5 & 1 & -3 & | & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 & -1 & | & 0 \\ \boxed{-2} & 2 & 1 & 2 & | & -1 \\ \boxed{1} & 0 & -1 & 3 & | & -1 \\ \boxed{3} & 2 & 1 & 0 & | & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 & -1 & | & 0 \\ \boxed{-2} & 2 & 1 & 2 & | & -1 \\ \boxed{1} & 0 & -1 & 3 & | & -1 \\ \boxed{3} & \boxed{1} & 0 & -2 & | & 0 \end{pmatrix}$$

backsub

$$x = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & 0 & 1 & \\ 3 & 1 & 0 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 1 & 0 & -1 \\ & 2 & 1 & 2 \\ & & -1 & 3 \\ & & & -2 \end{pmatrix}$$

Two steps:

$$Ly = b \quad y = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix}$$

$$Ux = y \quad x = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

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1. 2.3: 1,2,6,15

1. (2.3:1) Find $\|A\|_\infty$

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 5 & 1 \\ -1 & 2 & -3 \\ 1 & -7 & 0 \end{pmatrix}$$

- **ans:** (a) We find the max absolute row sum:

$$\|A\|_\infty = \max\{1+2,\ 4+3\} = 7$$

(b) We find the max absolute row sum:

$$\|A\|_\infty = \max\{1+5+1, 1+2+3, 1+7+0\} = 8$$

2. (2.3:6) Find the relative forward and backward errors, and error magnification factors. Verify that the factors are no bigger than the condition number.

$$(A|b) = \begin{pmatrix} 1 & 2 & | & 3 \\ 2 & 4.01 & | & 6.01 \end{pmatrix}$$

$$(a)\ x_c = \begin{pmatrix} -10 \\ 6 \end{pmatrix}, \qquad (b)\ x_c = \begin{pmatrix} -100 \\ 52 \end{pmatrix},$$

$$(c)\ x_c = \begin{pmatrix} -600 \\ 301 \end{pmatrix}, \qquad (d)\ x_c = \begin{pmatrix} -599 \\ 301 \end{pmatrix}$$

- **ans:** Easy to find the exact solution

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(a)

$$x_c = \begin{pmatrix} -10 \\ 6 \end{pmatrix},$$

$$f.r.e = \frac{\|x - x_c\|}{\|x\|} = \frac{11}{1} = 11$$

$$b.r.e = \frac{\|b - Ax_c\|}{\|b\|} = \frac{1.95}{6.01} = 0.3245$$

error magnification factor:

$$\beta = \frac{\text{relative forward error}}{\text{relative backward error}} \\ = 33.9026$$

(b)

$$x_c = \begin{pmatrix} -100 \\ 52 \end{pmatrix},$$

$$f.r.e = \frac{\|x - x_c\|}{\|x\|} = \frac{101}{1} = 101$$

$$b.r.e = \frac{\|b - Ax_c\|}{\|b\|} = \frac{2.51}{6.01} = 0.4176$$

error magnification factor:

$$\beta = \frac{\text{relative forward error}}{\text{relative backward error}} \\ = 241.8367$$

(c)

$$x_c = \begin{pmatrix} -600 \\ 301 \end{pmatrix},$$

$$f.r.e = \frac{\|x - x_c\|}{\|x\|} = \frac{601}{1} = 601$$

$$b.r.e = \frac{\|b - Ax_c\|}{\|b\|} = \frac{1}{6.01} = 0.1664$$

error magnification factor:

$$\begin{aligned}\beta &= \frac{\text{relative forward error}}{\text{relative backward error}} \\ &= 3612.01\end{aligned}$$

(d)

$$x_c = \begin{pmatrix} -599 \\ 301 \end{pmatrix}$$

$$f.r.e = \frac{\|x - x_c\|}{\|x\|} = \frac{600}{1} = 600$$

$$b.r.e = \frac{\|b - Ax_c\|}{\|b\|} = \frac{3}{6.01} = 0.499$$

error magnification factor:

$$\begin{aligned}\beta &= \frac{\text{relative forward error}}{\text{relative backward error}} \\ &= 1202\end{aligned}$$

$$\begin{aligned}\|A\|_\infty &= 6.01 \\ A^{-1} &= \begin{pmatrix} 401 & -200 \\ -200 & 100 \end{pmatrix} \\ \|A^{-1}\|_\infty &= 601 \\ \text{Cond}(A) &= \|A\|_\infty \|A^{-1}\|_\infty = 3612.01\end{aligned}$$

Yes. The condition is no less than the error magnification factors.

1. (2.4:a1) Solve the following system $(A|b)$

   (a) by GE without pivoting, (no row switching, no row multiplication)

   (b) by finding $A = LU$ and using it

   (c) by GE with partial pivoting, (row pivoting every step)

   (d) by finding $PA = LU$ and using it

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ -2 & 0 & 2 & 2 \\ 3 & -1 & -1 & 0 \end{array}\right)$$

- **ans:**

(a) $2r_1 + r_2$, $-3r_1 + r_3$:

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 0 & -6 & 6 & 12 \\ 0 & 8 & -7 & -15 \end{array}\right)$$

$(8/6)r_2 + r_3$:

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 5 \\ 0 & -6 & 6 & 12 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

(b) By above $2r_1 + r_2$, $-3r_1 + r_3$ and $(8/6)r_2 + r_3$:

$$\begin{aligned}L &= \begin{pmatrix} 1 & & \\ -2 & 1 & \\ 3 & -8/6 & 1 \end{pmatrix} \\ U &= \begin{pmatrix} 1 & -3 & 2 \\ 0 & -6 & 6 \\ 0 & 0 & 1 \end{pmatrix} \\ A &= LU\end{aligned}$$

Solving the system in two steps:

$$Ly = b \qquad y = \begin{pmatrix} 5 \\ 12 \\ 1 \end{pmatrix} \text{ (copy above)}$$

$$Ux = y \qquad x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

(c) Row switching: 1 3, (pivots: $1, -2, 3$)

$$\left(\begin{array}{ccc|cc} 3 & -1 & -1 & 0 & r_3 \\ -2 & 0 & 2 & 2 & r_2 \\ 1 & -3 & 2 & 5 & r_1 \end{array}\right)$$

$(2/3)r_1 + r_2$, $(-1/3)r_1 + r_3$:

$$\left(\begin{array}{ccc|cc} 3 & -1 & -1 & 0 & r_3 \\ \boxed{-2/3} & -0.66667 & 1.3333 & 2 & r_2 \\ \boxed{1/3} & -2.6667 & 2.3333 & 5 & r_1 \end{array}\right)$$

Row switching: 2 3, (pivots: $-0.66667, -2.6667$)

$$\left(\begin{array}{ccc|cc} 3 & -1 & -1 & 0 & r_3 \\ \boxed{1/3} & -2.6667 & 2.3333 & 5 & r_1 \\ \boxed{-2/3} & -0.66667 & 1.3333 & 2 & r_2 \end{array}\right)$$

$(-1/4)r_2 + r_3$:

$$\left(\begin{array}{ccc|cc} 3 & -1 & -1 & 0 & r_3 \\ \boxed{1/3} & -2.6667 & 2.3333 & 5 & r_1 \\ \boxed{-2/3} & \boxed{1/4} & 0.75 & 0.75 & \end{array}\right)$$

Backward substitution:

$$x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

(d) From the row index column above, we get

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

From the boxed numbers, we get

$$L = \begin{pmatrix} 1 & & \\ 1/3 & 1 & \\ -2/3 & 1/4 & 1 \end{pmatrix}$$

And the left over matrix above gives us

$$U = \begin{pmatrix} 3 & -1 & -1 \\ & -8/3 & 7/3 \\ & & 3/4 \end{pmatrix}$$

Three steps in solving $Ax = b$.

$$Pz = b \qquad z = Pb = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$$

$$Ly = b \qquad y = \begin{pmatrix} 0 \\ 5 \\ 3/4 \end{pmatrix} \text{ (copy above)}$$

$$Ux = y \qquad x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \text{ (copy above)}$$
