**M353 Hw 2** (S. Zhang) 1.2, 1.4, 1.5 2.1.

1. 1.2: a1, 2, 4, 5, 7

1. (1.2:a1) $g(x) = \frac{1}{2} + \frac{5}{8}x - \frac{x^2}{8}, \qquad x \in [-1, 2]$

   (a) Do 2 fixed point iteration $x = g(x)$ with $x_0 = 0$. What is the fixed point $x = r$?

   (b) Find local convergence at all fixed points.

   (c) Determine an upper bound for the number of iterations to reach $10^{-4}$ accuracy for the fix point iteration for initial $x_0 \in [-1, 2]$.

- **ans:**

(a)

$$\begin{aligned} p_0 &= 0 \\ p_1 &= g(p_0) = 0.5 \\ p_2 &= g(p_1) = 0.7812. \end{aligned}$$

They should converge to 1. Check if it is a fixed point:

$$g(1) = \frac{1}{2} + \frac{5}{8} - \frac{1}{8} = 1$$

Yes.

(b) We have to show $|g'(r)| < 1$.

$$\begin{aligned} g(x) &= \frac{1}{2} + \frac{5}{8}x - \frac{x^2}{8} \\ x &= \frac{1}{2} + \frac{5}{8}x - \frac{x^2}{8} \\ 8x &= 4 + 5x - x^2 \\ 0 &= x^2 + 3x - 4 = (x+4)(x-1) \\ r &= x = 1, -4. \end{aligned}$$

$$\begin{aligned} g'(x) &= \frac{5}{8} - \frac{x}{4} \\ g'(1) &= \frac{3}{8} \\ g'(-4) &= \frac{13}{8} \end{aligned}$$

So, the fixed point iteration is locally convergent for $r = 1$, but not for $r = -4$.

(c) By the theorem, for the fixed point $x$, we have

$$|x - p_n| \le K^n |x - p_0|,$$

where

$$\begin{aligned} K &= \max_{-1 \le x \le 2} |g'(x)| \\ &= \max_{-1 \le x \le 2} |\frac{5}{8} - \frac{x}{4}| \\ &= \max\{\frac{7}{8}, \frac{3}{8}\} = \frac{7}{8} = 0.875. \end{aligned}$$

To achieve,

$$|x - p_n| \le 10^{-4}$$

we can make

$$K^n |x - p_0| \le 10^{-4}$$

Since $x = 1$ is the fixed point and $K = 0.875$, we have, for $p_0 = 0$,

$$\begin{aligned} 0.875^n |1 - 0| &\le 10^{-4} \\ n &\ge \frac{\ln 10^{-4}}{\ln 0.875} = 68.97 \\ n &= 69. \end{aligned}$$

(d) For complete analysis (not required) here, we have to show two things: one, the function values of $g(x)$ is also between $-1$ and 2 when $x$ is between $-1$ and 2; two, the derivative $g'(x)$ is between $-K$ and $K$ for some $K < 1$ when $x$ is between $-1$ and 2.
(1) We need to find the minimal and maximal value of $g(x)$ on $[-1, 2]$. In calculus, we need to check the $g(x)$ values at the end points and all critical points.

$$\begin{aligned} g'(x) &= \frac{5}{8} - \frac{x}{4} \\ g'(x) &= 0 \quad \Rightarrow x = \frac{5}{2} \text{no critical point inside} \\ g(-1) &= -0.25 \\ g(2) &= 1.25 \end{aligned}$$

Therefore

$$\begin{aligned} -0.25 \le g(x) \le 1.25, &\quad x \in [-1, 2] \\ -1 \le g(x) \le 2, &\quad x \in [-1, 2] \\ g(x) \in [-1, 2] &\quad x \in [-1, 2]. \end{aligned}$$

(2) We need to find the minimal and maximal value of $f(x) = g'(x)$ on $[-1, 2]$. In calculus, we need to check the values at the end points and all critical points (when $f'(x) = g''(x) = 0$).

$$\begin{aligned} f'(x) &= g''(x) = -\frac{1}{4} \\ f'(x) &= 0 \quad \Rightarrow \text{no critical point} \\ f(-1) &= g'(-1) = 0.875 \\ f(2) &= -0.125 \end{aligned}$$

Therefore

$$|g'(x)| \le 0.875$$

So we let

$$K = 0.875 < 1$$

By (1) and (2), $g(x)$ has a unique fixed point in $[-1, 2]$.

2. (1.2:7) Find the two roots of

$$f(x) = 2x^2 + x - 1 = 0$$

isolate the $x^2$ term and solve for $x$ to find two candidates for $g(x)$. Which of the roots will be found by the two fixed-point iterations?

- **ans:**

$$2x^2 + x - 1 = 0$$

$$(2x-1)(x+1) = 0, \ x = 1/2, -1$$

By

$$2x^2 + x - 1 = 0$$

we get

$$x^2 = \frac{1-x}{2}$$

Taking square root, we get two functions:

(1)

$$x = g(x) = \sqrt{\frac{1-x}{2}}$$

$$g'(x) = -\frac{1}{2\sqrt{2-x}}$$
$$g'(\frac{1}{2}) = -\frac{1}{2}$$
$$g'(-1) = -\frac{1}{4}$$

Because $|g'(x)| < 1$, the fixed-point iteration are locally convergent to both roots, by theory. Let us try both:

$$x_0 = \frac{1}{2}, \ x_1 = g(x_0) = \frac{1}{2}, \ ...$$

yes, converges.

$$x_0 = -1, \ x_1 = g(x_0) = 1, \ x_2 = 0.707, \ ...$$

yes, converges, but it does not converge to $-1$. It converges to the other root $1/2$. This is because of the minus sign.

(2)

$$x = g(x) = -\sqrt{\frac{1-x}{2}}$$

$$g'(x) = \frac{1}{2\sqrt{2-x}}$$
$$g'(\frac{1}{2}) = \frac{1}{2}$$
$$g'(-1) = \frac{1}{4}$$

Because $|g'(x)| < 1$, the fixed-point iteration are locally convergent to both roots, by theory. Let us try both:

$$x_0 = \frac{1}{2}, \ x_1 = g(x_0) = -\frac{1}{2}, \ x_2 = -0.866, ...$$

yes, converges, but it does not converge to $1/2$. It converges to the other root $-1$. This is because of the minus sign..

$$x_0 = -1, \ x_1 = g(x_0) = -1, \ x_2 = -1, \ ...$$

yes, it converges to $-1$.

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1. 1.4: 2,4,6,7

1. (1.4:2) Do 2 steps of the Newton's method, $x_0 = 1$.

$$(a) \ x^3 + x^2 - 1 = 0$$
$$(b) \ x^2 + \frac{1}{x+1} - 3x = 0$$
$$(c) \ 5x - 10 = 0$$

- **ans:**

(a)

$$f'(x) = 3x^2 + 2x$$

$$x_1 = x_0 - f(x_0)/f'(x_0)$$

$$\begin{aligned} x_0 &= 1, & f(x_0) &= 1 & f'(x_0) &= 5 \\ x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 0.8, & f(x_1) &= 0.152 & f'(x_1) &= 3.52 \\ x_2 &= 0.7568 \end{aligned}$$

(b)

$$f'(x) = 2x - \frac{1}{(x+1)^2} - 3$$

$$x_1 = x_0 - f(x_0)/f'(x_0)$$

$$\begin{aligned} x_0 &= 1, & f(x_0) &= -1.5 & f'(x_0) &= -1.25 \\ x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= -0.2, & f(x_1) &= 1.89 & f'(x_1) &= -4.96 \\ x_2 &= 0.1809 \end{aligned}$$

(c)

$$f'(x) = 5$$

$$x_1 = x_0 - f(x_0)/f'(x_0)$$

$$
\begin{aligned}
x_0 &= 1, & f(x_0) &= -5 & f'(x_0) &= 5\\
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)}\\
&= 2, & f(x_1) &= 0 & f'(x_1) &= 5\\
x_2 &= 2
\end{aligned}
$$

We got the exact solution in one step.

2. (1.4:4b) Find convergence order and the rate

$$x^3 - x^2 - 5x - 3 = -0,\ r = -1, 3$$

- **ans:**

$$
\begin{aligned}
f'(x) &= 3x^2 - 2x - 5\\
f''(x) &= 6x - 2
\end{aligned}
$$

$$
\begin{aligned}
r &= -1\\
f'(r) &= 0\\
f''(r) &= -8\\
\frac{e_{i+1}}{e_i} &\to S = \frac{m-1}{m} = \frac{2-1}{2} = \frac{1}{2}
\end{aligned}
$$

We have a linear convergence.

$$
\begin{aligned}
r &= 3\\
f'(r) &= 16 \neq 0\\
f''(r) &= 16\\
\frac{e_{i+1}}{e_i^2} &\to M = \left|\frac{f''(r)}{2f'(r)}\right| = \frac{1}{2}
\end{aligned}
$$

We have a quadratic convergence.

3. (1.4:6) Sketch a function and initial guess for which Newton's method diverges

- **ans:**

$$
\begin{aligned}
f(x) &= \arctan x\\
f'(x) &= \frac{1}{1+x^2}
\end{aligned}
$$

$$x_0 = 1.5$$

$$
\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = -1.694\\
x_2 &= 2.3211\\
x_3 &= -5.114
\end{aligned}
$$

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1. (old) 1.5: 1ab, 2ab, 4

   (new) 1.5: 1bc, 2bc, a1-a2

1. (1.5:2b) Apply two steps of the false method for the root of the function:

$$e^x + x = 7,\ x_0 = 1,\ x_1 = 2$$

- **ans:**

$$f(x) = e^x + x - 7$$

We trap the root, by having two different function signs at the two ends.

$$c = b - f(b)\frac{b-a}{f(b)-f(a)}$$

| $i$ | $a, f(a)$ | $c, f(c)$ | $b, f(b)$ |
|---|---|---|---|
| 1 | $1_{-3.28}$ | $1.6_{-0.44}$ | $2.0000_{2.38}$ |
| 2 | $1.6_{-0.44}$ | $1.7423_{0.4525}$ | $2.0000_{2.38}$ |
| 3 | $1.6_{-0.44}$ | | $1.7423_{0.4525}$ |

2. (1.5:a1) For finding the root of the function:

$$f(x) = x^2 - 5x - 11$$

(a) Do 3 steps of the Newton's method, $p_0 = 6$. Find the errors and use the data to show the method is a second-order one.

(b) Do 3 steps of the secant method, $p_0 = 0$, $p_1 = 8$.

(c) Do 3 steps of the false position method, given the initial interval $[0, 8]$.

- **ans:**

(a) Each iteration would double the number of correct digits, for a second order iteration. Exact solution

$$x = 6.6533$$

$$p_1 = p_0 - f(p_0)/f'(p_0)$$
$$f'(x) = 2x - 5$$

| $k$ | $p$ | $f(p)$ | $f'(p)$ | error |
|---|---|---|---|---|
| 1 | 6.0000 | −5.0000 | 7.0000 | 0.6533 |
| 2 | 6.7143 | 0.5102 | 8.4286 | −0.0610 |
| 3 | 6.6538 | 0.0037 | 8.3075 | −0.0004 |

The errors are reduced by 1/10 and 1/100. if we do one more iteration, the error is to be reduced by 1/10000. If we do a few more Newton's iterations, we get the exact solution

$$x = 6.65331173.$$

(b)

$$p_2 = p_1 - f(p_1)\frac{p_1 - p_0}{f(p_1) - f(p_0)}$$

| $k$ | $p$ | $f(p)$ |
|---|---|---|
| 0 | 0.0000 | −11.0000 |
| 1 | 8.0000 | 13.0000 |
| 2 | 3.6667 | −15.8889 |
| 3 | 6.0500 | −4.6475 |
| 4 | 7.0353 | 3.3193 |

(c)

$$c = b - f(b)\frac{b-a}{f(b) - f(a)}$$

| $i$ | $a, f(a)$ | $c, f(c)$ | $b, f(b)$ |
|---|---|---|---|
| 1 | $0.0000_{-11.0000}$ | $3.6667_{-15.8889}$ | $8.0000_{13.0000}$ |
| 2 | $3.6667_{-15.8889}$ | $6.0500_{-4.6475}$ | $8.0000_{13.0000}$ |
| 3 | $6.0500_{-4.6475}$ | $6.5635_{-0.7377}$ | $8.0000_{13.0000}$ |

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1. 2.1:2, 4 a1-a5

1. (2.1:2a) Use Gaussian elimination without pivoting and backward substitution to solve the system $Ax = b$ where $(A|b) =$

$$\begin{pmatrix} 2 & -2 & -1 & | & -2 \\ 4 & 1 & -2 & | & 1 \\ -2 & 1 & -1 & | & -3 \end{pmatrix}$$

- **ans:** All row operations must be exactly as shown here.

$$(A|b) \to (U|c)$$

$-2r_1 + r_2 \to r_2$, $r_1 + r_3 \to r_3$:

$$\begin{pmatrix} 2 & -2 & -1 & | & -2 \\ & 5 & 0 & | & 5 \\ & -1 & -2 & | & -5 \end{pmatrix}$$

$(1/5)r_2 + r_3 \to r_3$:

$$\begin{pmatrix} 2 & -2 & -1 & | & -2 \\ & 5 & 0 & | & 5 \\ & & -2 & | & -4 \end{pmatrix}$$

Backward substitution, starting from equation 3.

$$x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

2. (2.1:4a) Use Gaussian elimination without pivoting and backward substitution to solve the system $Ax = b$ where $(A|b) =$

$$\begin{pmatrix} 3 & -4 & -2 & | & 3 \\ 6 & -6 & 1 & | & 2 \\ -3 & 8 & 2 & | & -1 \end{pmatrix}$$

- **ans:**

All row operations must be exactly as shown here.

$$(A|b) \to (U|c)$$

$(-2)r_1 + r_2 \to r_2$, $(1)r_1 + r_3 \to r_3$:

$$\begin{pmatrix} 3 & -4 & -2 & | & 3 \\ & 2 & 5 & | & -4 \\ & 4 & 0 & | & 2 \end{pmatrix}$$

$(-2)r_2 + r_3 \to r_3$:

$$\begin{pmatrix} 3 & -4 & -2 & | & 3 \\ & 2 & 5 & | & -4 \\ & & -10 & | & 10 \end{pmatrix}$$

Backward substitution, starting from equation 3.

$$x = \begin{pmatrix} 1 \\ 1/2 \\ -1 \end{pmatrix}$$

3. (2.1:a1) Find $A^{-1}$ by elementary row operations.

$$A = \begin{pmatrix} 1 & -3 & 2 \\ -1 & 0 & 2 \\ 1 & -1 & -1 \end{pmatrix}$$

- **ans:**

$$\left(\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & & \\ -1 & 0 & 2 & & 1 & \\ 1 & -1 & -1 & & & 1 \end{array}\right)$$

$$\overset{r_1+r_2}{\longrightarrow}\left(\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & & \\ & -3 & 4 & 1 & 1 & \\ 1 & -1 & -1 & & & 1 \end{array}\right)$$

$$\overset{-r_1+r_3}{\longrightarrow}\left(\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & & \\ & -3 & 4 & 1 & 1 & \\ & 2 & -3 & -1 & & 1 \end{array}\right)$$

$$\overset{(-1/3)r_2}{\longrightarrow}\left(\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & & \\ & 1 & -4/3 & -1/3 & -1/3 & \\ & 2 & -3 & -1 & & 1 \end{array}\right)$$

$$\overset{3r_2+r_1}{\longrightarrow}\left(\begin{array}{ccc|ccc} 1 & & -2 & 0 & -1 & \\ & 1 & -4/3 & -1/3 & -1/3 & \\ & 2 & -3 & -1 & & 1 \end{array}\right)$$

$$\overset{-2r_2+r_3}{\longrightarrow}\left(\begin{array}{ccc|ccc} 1 & & -2 & 0 & -1 & \\ & 1 & -4/3 & -1/3 & -1/3 & \\ & & -1/3 & -1/3 & 2/3 & 1 \end{array}\right)$$

$$\overset{(-3)r_3}{\longrightarrow}\left(\begin{array}{ccc|ccc} 1 & & -2 & 0 & -1 & \\ & 1 & -4/3 & -1/3 & -1/3 & \\ & & 1 & 1 & -2 & -3 \end{array}\right)$$

$$\overset{(4/3)r_3+r_2}{\longrightarrow}\left(\begin{array}{ccc|ccc} 1 & & -2 & 0 & -1 & \\ & 1 & & 1 & -3 & -4 \\ & & 1 & 1 & -2 & -3 \end{array}\right)$$

$$\overset{2r_3+r_1}{\longrightarrow}\left(\begin{array}{ccc|ccc} 1 & & -2 & 2 & -5 & -6 \\ & 1 & & 1 & -3 & -4 \\ & & 1 & 1 & -2 & -3 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 2 & -5 & -6 \\ 1 & -3 & -4 \\ 1 & -2 & -3 \end{pmatrix}$$

4. (2.1:a3) Find $\det(A)$ by direct expansions. Then again, by reducing $A$ to an upper triangular matrix.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 7 \\ -6 & -12 & -22 \end{pmatrix}$$

- **ans:** Expand by column 1:

$$\det A = \det\begin{pmatrix} 4 & 7 \\ -12 & -22 \end{pmatrix} - 2\det\begin{pmatrix} 1 & 2 \\ -12 & -22 \end{pmatrix}$$
$$- 6\det\begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix} = -4 - 4 + 6 = -2$$

$$A \overset{-2r_1+r_2}{\longrightarrow} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ -6 & -12 & -22 \end{pmatrix}$$

$$\overset{6r_1+r_3}{\longrightarrow} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ & -6 & -10 \end{pmatrix}$$

$$\overset{3r_2+r_3}{\longrightarrow} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ & & -1 \end{pmatrix}$$

$$\det A = (1)(2)(-1) = -2$$

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