M353 Tutorial/Lab 11(b) (S. Zhang) .

1.
$$A=\begin{pmatrix}7 & 1 & -1\\ 3 & 14 & 1\\ 5 & 3 & 28\end{pmatrix}, b=\begin{pmatrix}1\\2\\3\end{pmatrix}$$

By MATLAB, find x_4 in the Jacobi iteration. Here x_0 is the zero vector.

Find the error reduction rate $r = \|R_j\|_\infty$, and the errors in the infinity norm $\|x-x_0\|_\infty$, $\|x-x_4\|_\infty$, and compare $\|x-x_4\|_\infty/\|x-x_0\|_\infty$ and r^4.

```
format long
A=[7 1 -1;3 14 1; 5 3 28];b=[1 2 3]';
D=diag(diag(A));  A1=A-D; x=0*b;
x= inv(D)*(b-A1*x),  x= inv(D)*(b-A1*x)
x= inv(D)*(b-A1*x),  x= inv(D)*(b-A1*x)
Rj=inv(D)*A1, rate=norm(Rj,inf),
sol=inv(A)*b;
e=[norm(sol-x,inf)  norm(sol,inf)]
[e(1)/e(2) rate^4]

0.000078057473813   0.0066
```

2. (e304)
$$A=\begin{pmatrix}3 & 1 & 0\\ -1 & 3 & 2\\ 0 & 0 & 4\end{pmatrix}, b=\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

By MATLAB, find x_4 in the Jacobi iteration. Here x_0 is the zero vector.

Find the error reduction rate $r = \|R_j\|_\infty$, and the errors in the infinity norm $\|x-x_0\|_\infty$, $\|x-x_4\|_\infty$, and compare $\|x-x_4\|_\infty/\|x-x_0\|_\infty$ and r^4.

M353 Tutorial/Lab 12(c) (S. Zhang) .

1.
$$A=\begin{pmatrix}7 & 1 & -1\\ 3 & 14 & 1\\ 5 & 3 & 28\end{pmatrix}, b=\begin{pmatrix}1\\2\\3\end{pmatrix}$$

By MATLAB, find x_4 in the Gauss-Seidel iteration. Here x_0 is the zero vector.

Find the error reduction rate $r = \|R_{gs}\|_\infty$, and the errors in the infinity norm $\|x-x_0\|_\infty$, $\|x-x_4\|_\infty$, and compare $\|x-x_4\|_\infty/\|x-x_0\|_\infty$ and r^4.

```
format long
A=[7 1 -1;3 14 1; 5 3 28];b=[1 2 3]';
D=diag(diag(A));
U=triu(A,1); L=A-U; x=0*b;
x=inv(L)*(b-U*x), x=inv(L)*(b-U*x)
x=inv(L)*(b-U*x), x=inv(L)*(b-U*x)
Rgs=inv(L)*U, rate=norm(Rgs,inf),
sol=inv(A)*b;
e=[norm(sol-x,inf)  norm(sol,inf)]
[e(1)/e(2) rate^4]

0.000007483502655   0.006663890045814
```

2. (e305)
$$A=\begin{pmatrix}2 & -1 & 1\\ -1 & 2 & 0\\ 1 & 0 & 3\end{pmatrix}, b=\begin{pmatrix}-1\\1\\-1\end{pmatrix}$$

By MATLAB, find x_4 in the Gauss-Seidel iteration. Here x_0 is the zero vector.

Find the error reduction rate $r = \|R_{gs}\|_\infty$, and the errors in the infinity norm $\|x-x_0\|_\infty$, $\|x-x_4\|_\infty$, and compare $\|x-x_4\|_\infty/\|x-x_0\|_\infty$ and r^4.

M353 Tutorial/Lab 13(d) (S. Zhang) .

1. Do conjugate gradient iteration with $x_0 = [0\ 0]$.
$$A=\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix},\ b=\begin{pmatrix}0\\1\end{pmatrix}$$

Output all intermediate results.

```
b=[0 1]'; x=[0 0]'; A=[1 1; 1 2];
d=b, r=b,

al=(r'*r)/(d'*A*d),
x = x + al * d
r1 = r  -  al* A *d
ba = (r1'*r1) /(r'*r)
d = r1  + ba * d
r=r1;

al=(r'*r)/(d'*A*d),
x = x + al * d
r1 = r  -  al* A *d
```

2. (e306) Do conjugate gradient iteration (3 steps)
$$A=\begin{pmatrix}1 & 1 & 2\\ 1 & 2 & 3\\ 2 & 3 & 9\end{pmatrix},\ b=\begin{pmatrix}0\\7\\-6\end{pmatrix},\ x_0=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Output all intermediate results.

M353 Tutorial/Lab 14(e) (S. Zhang) .

1. Find the least-squares solution

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & -2 \\ 1 & 2 & 5 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix}$$

Find the residual $\|b - Ax\|_2$.

```
A=[0 1 1; 1 2 1; 1 -1 -2;1 2 5];
b=[1 2 3 5]';
method1_x=A\b, x=inv(A'*A)*A'*b
norm(b-A*x,2)
comapre_to_0_sol=norm(b,2)
   1.2060
```

2. (e307) Find the least-squares solution

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 0 \\ 1 & 4 & 9 \\ 1 & 2 & 5 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

Find the residual $\|b - Ax\|_2$.

M353 Tutorial/Lab 15(f) (S. Zhang) .

1. Find the least-squares solution

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Find the length of the solution $\|x\|_2$. $x_0 = [0\ 0\ 0\ 0\ 1]$ is another solution. Is $\|x\|_2$ smaller than $\|x_0\|_2 = 1$?

```
A=[1 0 1 0 1; 1 2 3 4 0]; b=[1;0];
   x=A'*inv(A*A')*b,
 check=b-A*x, x0=[0 0 0 0 1]',
 checking=b-A*x0
 length_x=norm(x,2),
 length_x0=norm(x0,2)
  0.6367
```

2. (e308) Find the least-squares solution

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Find the length of the solution $\|x\|_2$. $x_0 = [0\ 1\ 0\ 0\ 0]$ is another solution. Is $\|x\|_2$ smaller than $\|x_0\|_2 = 1$?

M353 Tutorial/Lab 16(g) (S. Zhang) .

1. Find the least-squares fit at the points $x_i = 0, 1, 2, 3$ of $f(x) = 1 + x^2$ by (a)$y = Ax^3 + B$, (b) $y = Ce^{Dx}$, (c) $y = Ax + B$,. Find $E_2(f)$ for each case. Which one is better?

```
x=0:3; x=x',
f=inline('1+x.^2'); y=f(x);
A=[x.^3 ones(4,1)]; b=y;  s=A\b,
E2=norm(b-A*s,2)/sqrt(4),
A=[ones(4,1) x]; b=log(y); s=A\b
g=exp(s(1)+s(2)*x),
E2=norm(g-y,2)/sqrt(4)
A=[x ones(4,1)]; b=y; s=A\b
E2=norm(b-A*s,2)/sqrt(4)
s =
    3.0000
   -0.0000
E2 =
    1
```

2. (e309) Find the least-squares fit at the points $x_i = 1, 2, 3, 4$ of $f(x) = 2.3 - 3.7x^3$ by $y = A + Bx$ Find $E_2(f)$.

M353 Tutorial/Lab 17(h) (S. Zhang) .

1. Find Least sqaure $P_3(x)$ which fits the data

i	0	1	2	3	4	5
x_i	2	3	4	5	6	7
y_i	0	1	2	2.5	2	0

Output the coefficients of $P_3(x)$, and the values of $P_3(x)$ at $x = 0, 1, 2, 3$.

```
x=(2:7)';
y=[0  1   2  2.5 2 0 ]';
m=size(x,1);  %%
A=[ones(m,1) x x.^2 x.^3];
p=inv(A'*A)*A'*y
p3=rot90(p')
val=polyval(p3,0:3)
```

2. (e310) Find Least sqaure $P_4(x)$ which fits the data

i	0	1	2	3	4	5
x_i	0.2	0.8	1.5	2	2.2	2.8
y_i	0	1	2	2.5	1.7	0

Output the 5 coefficients of $P_4(x)$, and the values of $P_4(0.8)$, and $P_4(1)$.

M353 Tutorial/Lab 18(i) (S. Zhang) .

1. Given function

$$f(x) = \sqrt{5x^2 - 1}$$

approximate $f'(x)$ by the central difference method with $h = 0.1$, $h = 0.01$ at $x_0 = 1$. Then extrapolate the data to get the best $f'(x_0)$.

```
 f=inline('sqrt(5*x.^2-1)'); x0=1;
h=0.1;  d1=(f(x0+h)-f(x0-h))/(2*h),
h=h/10; d2=(f(x0+h)-f(x0-h))/(2*h),
extrapolation=(10^2*d2-d1)/(10^2-1)
% To check
d=inline('(5*x.^2-1).^-.5*(5*x)');
err=d(x0)-[d1 d2 extrapolation]'
```

2. (e311) Given function

$$f(x) = \sqrt{\tan(\sqrt{2x + 0.1})}$$

approximate $f'(x)$ by the central difference method with $h = 0.1$, $h = 0.05$ at $x_0 = 0.1$. Then extrapolate the data to get the best $f'(x_0)$.

M353 Tutorial/Lab 19(j) (S. Zhang) .

1. Given function

$$f(x) = 5x^4 - 1$$

approximate $f''(x)$ by the central difference method with $h = 0.1$, $h = 0.02$ at $x_0 = 1$. Then extrapolate the data to get the best $f''(x_0)$.

```
 f=inline('5*x.^4-1'); x0=1;
h=0.1; d1=(f(x0+h)-2*f(x0)+f(x0-h))/(h^2),
h=h/5; d2=(f(x0+h)-2*f(x0)+f(x0-h))/(h^2),
extrapolation=(5^2*d2-d1)/(5^2-1)
% To check
d=inline('60*x.^2');
err=d(x0)-[d1 d2 extrapolation]'
```

2. (e312) Given function

$$f(x) = \sqrt{\tan(\sqrt{2x + 0.1})}$$

approximate $f''(x)$ by the central difference method with $h = 0.1$, $h = 0.05$ at $x_0 = 0.1$. Then extrapolate the data to get the best $f''(x_0)$.

M353 Tutorial/Lab 20(k) (S. Zhang) .

1. Compute $\int_0^{0.4} \cos 4x\, dx$. by the trapezoidal rule with $m = 5$ and $m = 20$. Extrapolate the data. Find three errors.

```
clc; clear
f=inline('cos(4*x)','x');
a=0;b=0.4; int=(sin(4*b)-sin(4*a))/4;
m=5;h=(b-a)/m; M=0;
 for x=a+h:h:b; M=M+f(x-h)/2+f(x)/2; end;
 T1=M*h,

% 4 times!
 n=4;
       h=h/n; M=0;
 for x=a+h:h:b; M=M+f(x-h)/2+f(x)/2; end;
 T2=M*h
 extra=(n^2*T2-T1)/(n^2-1),
 [int-T1 int-T2 int-extra]
```

2. (e313) Compute $\int_0^{0.5} \sin(4 + 4x)\, dx$. by the trapezoidal rule with $m = 10$ and $m = 20$. Extrapolate the data. Find three errors.

M353 Tutorial/Lab 21(l) (S. Zhang) .

1. Compute $\int_{-2}^{2.2} 5x^4\, dx$. by the Simpson's rule with $m = 2$ and $m = 4$. Extrapolate the data. Find three errors.

```
clc; clear; format short
f=inline('5*x^4','x');
a=-2;b=2.4;
int=b^5-a^5;
m=4;
h=(b-a)/(2*m); M=0;
for x=a+h:2*h:b;
  M=M+f(x-h)+4*f(x)+f(x+h);
end; S1=M*h/3,

h=h/2; M=0;
for x=a+h:2*h:b;
  M=M+f(x-h)+4*f(x)+f(x+h);
end; S2=M*h/3

extra=(2^4*S2-S1)/(2^4-1),
format long
err=[int-S1 int-S2 int-extra]'
```

2. (e314) Compute $\int_{-1}^{1.2} 25x^8\, dx$. by the Simpson's rule with $m = 2$ and $m = 8$. (Warning, 4 times big.) Extrapolate the data. Find three errors.

M353 Tutorial/Lab 22(m) (S. Zhang) .

1. Find the Romberg integral R_{33}

$$\int_{0.04}^{1} \sqrt{1 + x^2}\, dx$$

$$T_{M=1} = h[\frac{f_0}{2} + \frac{f_1}{2}]$$
$$T_{M=2} = \frac{T_{M=1}}{2} + hf_1$$
$$T_{M=4} = \frac{T_{M=2}}{2} + h[f_1 + f_3]$$

$$R_{11} = T_{M=1}$$
$$R_{21} = T_{M=2}$$
$$R_{31} = T_{M=4}$$
$$R_{22} = \frac{4R_{21} - R_{11}}{3}$$
$$R_{32} = \frac{4R_{31} - R_{21}}{3}$$
$$R_{33} = \frac{16R_{32} - R_{22}}{16 - 1}$$

```
clc; format long
f=inline('sqrt(1+x^2)','x');
a=0.04;b=1;
h=(b-a); M=0;
for x=a+h:h:b;
  M=M+f(x-h)/2+f(x)/2; end; R11=M*h;
h=h/2; M=0; for x=a+h:h:b;
  M=M+f(x-h)/2+f(x)/2; end; R21=M*h;
h=h/2; M=0; for x=a+h:h:b;
  M=M+f(x-h)/2+f(x)/2; end; R31=M*h;
R1=[R11 R21 R31]'
R2=[ (4*R21-R11)/3 (4*R31-R21)/3]
 R3=[ (16*R2(2)-R2(1))/15 ]
```

2. (e315) Find the Romberg integral R_{33}

$$\int_{0.04}^{1.1} e^{1+x^2}\,dx$$

M353 Tutorial/Lab 23(n) (S. Zhang) .

1. Compute the integral by adaptive quadrature with tolerance 0.1

$$\int_0^1 8x^3\,dx.$$

(a) On $(0,1)$,

$$T_{h=1} = 4,\ T_{h=1/2} = 2.5$$
$$tol = 3 \cdot TOL \cdot \frac{b_1 - a_1}{b - a} = 0.3$$
$$T_{h=1} - T_{h=1/2} = 1.5$$

Not accepted.

(b) On $(0,1/2)$,

$$T_{h=1/2} = 0.25,\ T_{h=1/4} = 0.1562$$
$$tol = 3 \cdot TOL \cdot \frac{b_1 - a_1}{b - a} = 0.15$$
$$T_{2h} - T_h = 0.0938$$

Accepted.

(c) On $(1/2,1)$,

$$T_{h=1/2} = 2.25,\ T_{h=1/4} = 1.9688$$
$$tol = 3 \cdot TOL \cdot \frac{b_1 - a_1}{b - a} = 0.15$$
$$T_{2h} - T_h = 0.2812$$

Not accepted.

(d) On $(1/2,3/4)$,

$$T_{h=1/4} = 0.5469,\ T_{h=1/8} = 0.5176$$
$$tol = 3 \cdot TOL \cdot \frac{b_1 - a_1}{b - a} = 0.075$$
$$T_{2h} - T_h = 0.0293$$

Accepted.

(e) On $(3/4,1)$,

$$T_{h=1/4} = 1.4219,\ T_{h=1/8} = 1.3809$$
$$tol = 3 \cdot TOL \cdot \frac{b_1 - a_1}{b - a} = 0.075$$
$$T_{2h} - T_h = 0.0410$$

Accepted.

(f) Together

$$T = 0.1562 + 0.5176 + 1.3809 = 2.0547$$

Error

$$\int -T = 2 - 2.0547 = -.0547 < 0.1$$

```
 clc; format short
f=inline('8*x^3 ');
a=0; b=1; tol=0.1;
t=[a,b];
c=(t(1)+t(2))/2; h=t(2)-t(1);
T1=(f(t(1))+f(t(2)))*h/2;
T2=T1/2+f(c)*h/2;
s1=[T1 T2 T1-T2 3*tol*(t(2)-t(1))/(b-a)]'
%% not accepted

t1=[t(1) c];
t2=[c t(2)];

t=t1;
  c=(t(1)+t(2))/2; h=t(2)-t(1);
  T1=(f(t(1))+f(t(2)))*h/2;
```

```
  T2=T1/2+f(c)*h/2;
s21=[T1 T2 T1-T2 3*tol*(t(2)-t(1))/(b-a)]'
  %% accepted and save it
s=T2;

t=t2; c=(t(1)+t(2))/2;
  h=t(2)-t(1);
  T1=(f(t(1))+f(t(2)))*h/2;
  T2=T1/2+f(c)*h/2;
s22=[T1 T2 T1-T2 3*tol*(t(2)-t(1))/(b-a)]'
 %% not accepted

t1=[t(1) c];
t2=[c t(2)];

t=t1;
  c=(t(1)+t(2))/2; h=t(2)-t(1);
  T1=(f(t(1))+f(t(2)))*h/2;
  T2=T1/2+f(c)*h/2;
  s31=[T1 T2 T1-T2 3*tol*(t(2)-t(1))/(b-a)]'
  %% accepted and add it
 s=s+T2;

t=t2;
  c=(t(1)+t(2))/2; h=t(2)-t(1);
  T1=(f(t(1))+f(t(2)))*h/2; T2=T1/2+f(c)*h/2;
  s32=[T1 T2 T1-T2 3*tol*(t(2)-t(1))/(b-a)]'
  % accepted and add it
s=s+T2
err=2*(b^4-a^4)-s
```

2. (e316) Compute the integral by adaptive quadrature with tolerance 0.1

$$\int_0^1 8(x-1.1)^3\,dx.$$

Solution: By hand

(a) On $(0,1)$,

$$T_{h=1} = -5.3,\ T_{h=1/2} = -3.5$$
$$tol = 3\cdot TOL\cdot\frac{b_1-a_1}{b-a} = 0.3$$
$$T_{h=1} - T_{h=1/2} = -1.8$$

Not accepted.

(b) On $(0,1/2)$,

$$T_{h=1/2} = -3.09,\ T_{h=1/4} = -2.77$$
$$tol = 3\cdot TOL\cdot\frac{b_1-a_1}{b-a} = 0.15$$
$$T_{2h} - T_h = -0.31$$

Still not accepted.

(c) On $(0,1/4)$,

$$T_{h=1/4} = --1.9,\ T_{h=1/2} = -1.89$$
$$tol = 3\cdot TOL\cdot\frac{b_1-a_1}{b-a} = 0.075$$
$$T_{2h} - T_h = -0.045$$

Accepted.

(d) On $(1/4,1/2)$,

$$T_{h=1/4} = ---0.83,\ T_{h=1/2} = -0.79$$
$$tol = 3\cdot TOL\cdot\frac{b_1-a_1}{b-a} = 0.075$$
$$T_{2h} - T_h = -0.034$$

Accepted.

(e) On $(1/2,1)$,

$$T_{h=1/2} = -0.43,\ T_{h=1/4} = -0.30$$
$$tol = 3\cdot TOL\cdot\frac{b_1-a_1}{b-a} = 0.15$$
$$T_{2h} - T_h = -0.13$$

Accepted.

(f) Together

$$T = T_{(0,1/4)} + T_{(1/4,1/2)} + T_{(1/2,1)}$$
$$= -2.998$$

Error

$$\int -T = 0.0703 < 0.1$$

M353 Tutorial/Lab 24(o) (S. Zhang) .

1. Given exact soltuion

$$y = \sqrt{1+t+t^2}$$

Find $y(1)$ by Euler method with $h = 0.1$ and $h = 0.01$. Extrapolate the solutions for $y(1)$. Find errors.

$$y' = \frac{1+2t}{2y};\quad y(0) = 1.$$

```
clear; clc, format long
f=inline('(1+2*t)/(2*y)','t','y');
sol=inline('sqrt(1+t+t^2)','t');
h=0.1; y=1;
for t=h:h:1,
  y=y+h*f(t,y); end;
s1=[h y sol(t) sol(t)-y]', y1=y;

n=10;
```

```
h=h/n;
y=1;
for t=h:h:1,
  y=y+h*f(t,y); end;
s2=[h y sol(t) sol(t)-y]', y2=y;

 extr=(n*y2-y1)/(n-1);
ex=[extr sol(t)-extr]',
```

2. (e317) Given exact soltuion

$$y = \sqrt{1+t+t^2}$$

Find $y(2)$ by Euler method with $h = 0.1$ and $h = 0.02$. Extrapolate the solutions for $y(2)$. Find errors.

$$y' = \frac{1+2t}{2y}; \quad y(0) = 1.$$

M353 Tutorial/Lab 25(p) (S. Zhang) .

1. Given an initial value problem

$$x'' - x' - 2x = 2t, \quad 0 < t < 1$$
$$x(0) = 2, \quad x'(0) = -1.$$

and the exact solution

$$x = e^{-t} + \frac{1}{2}e^{2t} + \frac{1}{2} - t$$

Convert the equation to a system. Use Runge-Kutta method with step size $h = \frac{1}{4}$ and find the error.

```
clc; format long,   clear;
s=inline('exp(-t)+exp(2*t)/2+1/2-t');
f=inline('[y  y+2*x+2*t]','t','x','y');
h=1/4; t=0; Z=[2,-1];
 for i=1:4,
   k1=f(t,Z(1),Z(2));
  Z1=Z+k1*h/2;
   k2=f(t+h/2,Z1(1),Z1(2) );
  Z1=Z+k2*h/2;
   k3=f(t+h/2,Z1(1),Z1(2) );
  Z1=Z+k3*h  ;
   k4=f(t+h,  Z1(1),Z1(2) );
  Z=Z+h*(k1+2*k2+2*k3+k4)/6,
  t=t+h,
 end
 err=s(t)-Z(1)
```

2. (e318) Given an initial value problem

$$x'' - 4x = 8t, \quad 0 < t < 2$$
$$x(0) = 2, \quad x'(0) = -2.$$

and the exact solution

$$x = e^{-2t} + e^{2t} - 2t$$

Convert the equation to a system. Use Runge-Kutta method with step size $h = \frac{1}{8}$ and find the error at $t = 2$.

M353 Tutorial/Lab 26(q) (S. Zhang) .

1. Given an initial value problem

$$x'' - 4x = 8t, \quad 0 < t < 1$$
$$x(0) = 1, \quad x(1) = e^2 - 2.$$

and the exact solution

$$x = e^{2t} - 2t$$

Use finite difference method with step size $h = \frac{1}{4}$ and find the error at $t = \frac{1}{2}$.

```
clc; format long,   clear;
s=inline('exp(2*t)-2*t');

 h=1/4; a=0;b=1;
 xa=1; xb=exp(2)-2;
 n=(b-a)/h-1;

% grid:
 t=a+h:h:b-h;

% right hand side (use b again)
  b=h^2*(8*t');
 b(1)=b(1)-xa;
 b(n)=b(n)-xb,

% set up matrix
 for i=1:n-1;
  a(i,i)=-2-4*h^2;
  a(i,i+1)=1;
  a(i+1,i)=1;
 end;
 a(n,n)=-2-4*h^2,

 x=inv(a)*b
 er=s(1/2)-x((n+1)/2)


x =

   1.161903832951387
   1.739283624140621
   3.001484321365009


er =

  -0.021001795681575
```

2. (e319) Given an initial value problem

$$\begin{aligned} x'' - 4x &= 8t, \quad 0 < t < 2 \\ x(0) &= 2, \quad x(1) = e^2 + e^{-2} - 2. \end{aligned}$$

and the exact solution

$$x = e^{-2t} + e^{2t} - 2t$$

Use finite difference method with step size $h = \frac{1}{8}$ and find the error at $t = 1$.