M351 Study Guide Final (S. Zhang) .

1. Solve
2.55

$$y' + y = \begin{cases} 1 & x < 1 \\ -1 & x \geq 1 \end{cases} \quad y(0) = 1$$

- **ans:** The solution is done in two steps.

First, we solve

$$y' + y = 1, \; y(0) = 1$$

$$\begin{aligned} \mu &= e^{\int p} = e^{\int 1} = e^x \\ y &= \frac{1}{\mu} \int \mu q = e^{-x} \int e^x \\ &= e^{-x}(e^x + C) \end{aligned}$$

By $y(0) = 1$,

$$\begin{aligned} &1 = 1 + C, \; C = 0 \\ &y = e^{-x} e^x = 1 \end{aligned}$$

Then we evaluate y at $x = 1$, to get $y = 1$ (usually we get e or e^2 or like that.)

In the second step, we solve the differential equation again, but with new right hand side function and new initial condition:

$$y' + y = -1, \; y(1) = 1$$

$$\begin{aligned} \mu &= e^{\int p} = e^{\int 1} = e^x \\ y &= \frac{1}{\mu} \int \mu q = e^{-x} \int -e^x \\ &= e^{-x}(-e^x + C) \end{aligned}$$

By $y(1) = 1$,

$$\begin{aligned} &1 = -1 + e^{-1} C, \; C = 2e \\ &y = -1 + 2e e^{-x} \end{aligned}$$

Hence, combine them together, we get

$$y = \begin{cases} 1 & x < 1 \\ -1 + 2e e^{-x} & x \geq 1 \end{cases}$$

2. Construct a table for y', y'' and solution curve $y(x)$ shapes.
804.16 Find and classify critical points Sketch phase portrait (phase line + direction field). Sketchall typical solution curves on the graph of direction field.

$$y' = 2y - y^2.$$

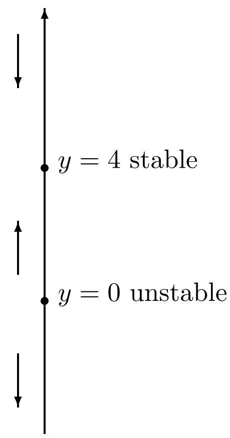
- **ans:**

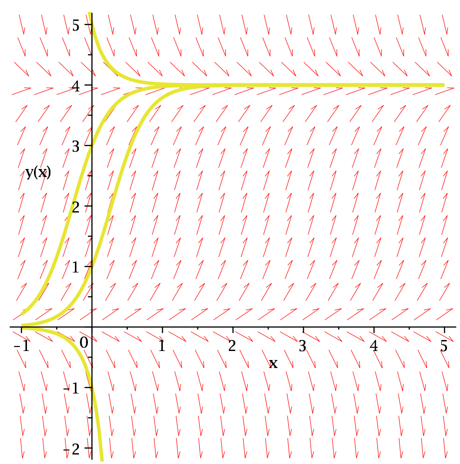
$$\begin{aligned} y' &= 4y - y^2 = y(4 - y) \\ y'' &= 4y' - 2yy' = (4 - 2y)y' \end{aligned}$$

By $y' = 0$, we have critical points $y = 0$ and $y = 4$.

For $y'' = 0$, we have an additional point, $y = 2$.

t		0		2		4	
y	$-$		$+$		$+$		$+$
$4 - y$	$+$		$+$		$+$		$-$
$y' = y(4 - y)$	$-$		$+$		$+$		$-$
up/down	$\searrow$		$\nearrow$		$\nearrow$		$\searrow$
$(4 - 2y)$	$+$		$+$		$-$		$-$
$y'' = (4 - 2y)y'$	$-$		$+$		$-$		$+$
shape	$\frown$		$\smile$		$\frown$		$\smile$






```
with(DEtools):
DEplot( diff(y(x),x)=y(x)*4-y(x)^2,y(x),
  x=-1..5,[[y(0)=1],[y(0)=3],[y(0)=5],
[y(0)=-1]],y=-2..5,
stepsize=.05);
```

3. Solve the IVP. Determine an interval for which the initial
4.15 value problem is certain to have a unique solution.

$$y' = \frac{2t}{2 - 2y}, \qquad y(3) = 2.$$

• **ans:** We solve the nonlinear equation first, then locate point t at which $y' = \pm\infty$.

$$\begin{aligned}(2-2y)dy &= 2tdt\\ \int(2-2y)dy &= \int(2t)dt\\ 2y-y^2 &= t^2+C\\ y(3)=2 \Rightarrow \qquad 4-4 &= 9+C,\ C=-9\\ 2y-y^2 &= t^2-9\end{aligned}$$

By the DE $y' = \pm\infty$ only when

$$2-2y=0 \quad \Rightarrow \quad y=1$$

When $y=1$, by above solution

$$\begin{aligned}2-1 &= t^2-9\\ t^2 &= 10\\ t &= \pm\sqrt{10}\end{aligned}$$

So the solution exists for $(-\infty, -\sqrt{10})$, $(-\sqrt{10}, \sqrt{10})$, $(\sqrt{10}, \infty)$. Since $y(3)=2$, the answer is $(-\sqrt{10}, \sqrt{10})$.

4. Determine if the differential equation is exact, and solve it
8.42 if it is.

$$(2y - \frac{1}{x} + \cos 3x)\frac{dy}{dx} + \frac{y}{x^2} - 4x^3 + 3y\sin 3x = 0$$

• **ans:** Rewrite the equation in the form $Mdx + Ndy = 0$:

$$(\frac{y}{x^2} - 4x^3 + 3y\sin 3x)dx + (2y - \frac{1}{x} + \cos 3x)dy = 0$$

Check:

$$\begin{aligned}M_y &= \frac{1}{x^2} + 3\sin 3x\\ N_x &= \frac{1}{x^2} - 3\sin 3x\end{aligned}$$

It is not an exact equation. We do not know how to solve it.

5. Solve the equation of $y' = f(y/x)$ form:
9.21

$$\frac{dy}{dx} = \frac{x+3y}{3x+y}$$

• **ans:** Let $y = ux$

$$\begin{aligned}u + xu' = y' &= \frac{x+3xu}{3x+xu}\\ u + xu' &= \frac{1+3u}{3+u}\end{aligned}$$

Separable

$$\begin{aligned}x\frac{du}{dx} &= \frac{1+3u}{3+u} - u\\ \frac{du}{\frac{1+3u}{3+u} - u} &= \frac{dx}{x}\\ \int \frac{(3+u)du}{1-u^2} &= \int \frac{dx}{x}\end{aligned}$$

We use partial fractions:

$$\begin{aligned}\frac{3+u}{1-u^2} &= \frac{A}{1-u} + \frac{B}{1+u}\\ 3+u &= A(1+u) + B(1-u)\\ (u=1) \Rightarrow\ A &= 2\\ (u=-1) \Rightarrow\ B &= 1\end{aligned}$$

So the above equation is

$$\int \frac{2du}{1-u} + \frac{du}{1+u} = \ln x + C$$

$$-\ln|1-u|^2 + \ln|1+u| = \ln x + C_1$$

$$\frac{|1+u|}{(1-u)^2} = Cx$$

$u = y/x$:

$$\frac{|1+y/x|}{(1-y/x)^2} = Cx$$

$$\frac{|x^2+xy|}{(x-y)^2} = Cx$$

$$C(x-y)^2 = (x+y)$$

6. A tank contains 100 liters of salt water with a concentration
810.46 of 2 g/liter. Pure water flows in at a rate of 2 liters/min, the well-stirred solution flowing out at the same rate. Find the time when the tank reaches 1 g/liter salt concentration.

• **ans:** Volume 100, mass $d(t)$.

In-rate:

$$\text{concentration} \cdot \text{speed} = 0 \cdot 2 = 0$$

Out-rate:

$$\begin{aligned}&\text{concentration} \cdot \text{speed}\\ =&\frac{\text{salt}}{\text{original volume} - \text{out volume} + \text{in volume}} \cdot \text{speed}\\ =&\frac{d(t)}{100+2t-2t} \cdot 2 = -\frac{d}{100}2\end{aligned}$$

$$\begin{aligned}d' &= \text{in-rate} - \text{out-rate}\\ &= 0 - \frac{d}{100}2,\ d(0) = 2\cdot 100\end{aligned}$$

Solve it as a separable equation or as a first order linear equation:

$$d = 200e^{-t/50}$$

$$d(t) = 1\cdot 100, \quad t = 50\ln 2 = 34.6$$

7. The time rate of change of an alligator population P in a
811.15 swamp is proportional to the square of P. The swamp contained a dozen alligators in 1988, two dozen in 1998. When will there be four dozen alligators in the swamp? What happens there after?

- **ans:**

$$
\begin{aligned}
&P' = kP^2, \quad -\frac{1}{P} = kt - C \\
&P = \frac{1}{C - kt} \\
&P(0) = 12,\ C = \frac{1}{12},\ P = \frac{12}{1 - 12kt} \\
&P(10) = 24,\ k = \frac{1}{240},\ P = \frac{240}{20 - t} \\
&P = 48, \quad t = 15 \\
&t \to 20, \quad P \to \infty
\end{aligned}
$$

8. Find the linear dependence by both finding nonzero solu-
11.44 tions and by Wrongskian:

$$y_1 = 1 + x^2,\ y_2 = x^2 - x,\ y_3 = x^2 + x$$

- **ans:** (1) By finding non zero solution:

$$
\begin{aligned}
c_1y_1 + c_2y_2 + c_3y_3 = 0 \\
c_1(1 + x^2) + c_2(x^2 - x) + c_3(x^2 + x) = 0
\end{aligned}
$$

Comparing coefficients:

$$
\begin{aligned}
x^2 : \quad & c_1 + c_2 + c_3 = 0 \\
x : \quad & -c_2 + c_3 = 0 \\
1 : \quad & c_1 = 0
\end{aligned}
$$

We have unique solution

$$c_1 = c_2 = c_3 = 0$$

Therefore, the three functions are linear independent. (If we can find one set of non-zero solutions, they are linearly dependent.)

(2) By Wronskian:

$$
\begin{aligned}
W &= \begin{vmatrix} 1 + x^2 & x^2 - x & x^2 + x \\ 2x & 2x - 1 & 2x + 1 \\ 2 & 2 & 2 \end{vmatrix} \\
&= (1 + x^2)\begin{vmatrix} 2x - 1 & 2x + 1 \\ 2 & 2 \end{vmatrix} - (x^2 - x)\begin{vmatrix} 2x & 2x + 1 \\ 2 & 2 \end{vmatrix} \\
&+ (x^2 + x)\begin{vmatrix} 2x & 2x - 1 \\ 2 & 2 \end{vmatrix} \\
&= (1 + x^2)(-4) - (x^2 - x)(-2) + (x^2 + x)(2) \\
&= -4 \neq 0
\end{aligned}
$$

So linearly independent.

9. Find the roots of characteristic equation and the general
16.28 solution:

(1) $4y'' - 4y' + y = 0$

(2) $(D - 1)^2(D^2 - 1)^2((D - 2)^2 - 9)((D - 1)^2 + 4)^2y = 0$

- **ans:** (1) Characteristic equation and roots

$$
\begin{aligned}
4r^2 - 4r + 1 = 0,\ (2r - 1)^2 = 0, \\
r = \frac{1}{2}, \frac{1}{2}
\end{aligned}
$$

Repeated roots.

The general solution is

$$y = Ae^{x/2} + Bxe^{x/2}$$

(2) Characteristic equation and roots

$$
\begin{aligned}
(r - 1)^2(D^2 - 1)^2((D - 2)^2 - 9)((D - 1)^2 + 4)^2 = 0, \\
r = 1, 1; \quad 1, -1, 1, -1; \quad -1, 5; \quad 1 \pm 2i, 1 \pm 2i
\end{aligned}
$$

The general solution is

$$
\begin{aligned}
y =& (A + Bx + Cx^2 + Dx^3)e^x + \\
&+ (F + Gx + Hx^2)e^{-x} + Ie^{5x} \\
&+ e^x((J + Kx)\cos 2x + (L + Mx)\sin 2x)
\end{aligned}
$$

10. Solve
20.51

$$y'' - 2y' - 3y = 3e^{2t}.$$

- **ans:** Characteristic equation is

$$r^2 - 2r - 3 = 0,\ r = -1, 3$$

$$y_H = c_1e^{-t} + c_2e^{3t}$$

$r \neq 2$,

$$y_P = Ae^{2t}$$

Plug y_P into DE:

$$A = -1$$

$$y = y_H + y_P = c_1e^{-t} + c_2e^{3t} - e^{2t}$$

11. Find the general solution of the <u>homogeneous</u> equation y_c
20.42 and <u>write down</u> the form for undetermined coefficients for y_p. (Do not solve for y_p.)

$$(D^2 - 9)(D + 3)(D^2 + 4)y = e^x \sin x - e^{-3x} + \cos 2x.$$

- **ans:**

$$r = 3, -3, -3, \pm 2i$$

$$
\begin{aligned}
y_c = y_H =& e^{-3x}(A + Bx) + e^{3x}(E) + ((G)\cos 2x + (I)\sin 2x. \\
y_p =& e^x(A\cos x + B\sin x) + (C)x^2e^{-3x} \\
&+ x(E\cos 2x + F\sin 2x)
\end{aligned}
$$

Note the extra x and x^2 above.

12. Solve
22.16

$$y''' + y' = \tan x$$

- **ans:** For $y_c = y_H$,

$$r^3 + r = 0,\ r = 0, \pm i$$

$$y_H = c_1 + c_2 \cos x + c_3 \sin x$$

For y_p by VP, (note that the method of undetermined coefficients won't work here as $\tan x$ is not one of those special functions),

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$$

satisfies eqautions:

$$\begin{aligned} u_1' y_1 + u_2' y_2 + u_3' y_3 &= 0 \\ u_1' y_1' + u_2' y_2' + u_3' y_3' &= 0 \\ u_1' y_1'' + u_2' y_2'' + u_3' y_3'' &= q \end{aligned}$$

Here we have

$$\begin{aligned} u_1' + u_2' \cos x + u_3' \sin x &= 0 \\ -u_2' \sin x + u_3' \cos x &= 0 \\ -u_2' \cos x - u_3' \sin x &= \tan x \end{aligned}$$

$(equation2) * \sin x + (equation3) * \cos x$:

$$\begin{aligned} -u_2' &= \tan x \cos x \\ u_2 &= \int - \tan x \cos x = - \int \sin x dx = \cos x \end{aligned}$$

$(equation2) * \cos x - (equation2) * \sin x$:

$$\begin{aligned} u_3' &= - \tan x \sin x \\ u_3 &= \int - \frac{\sin^2 x}{\cos x} \\ &= \int \frac{\cos^2 x - 1}{\cos x} = \int (\cos x - \sec x) \\ &= \int \cos x dx - \int \frac{\sec x(\sec x + \tan x)}{\sec x(\sec x + \tan x)} dx \\ &= \sin x - \ln(\sec x + \tan x) \end{aligned}$$

By $(equation1)$:

$$\begin{aligned} u_1' &= -u_2' \cos x - u_3' \sin x \\ &= \tan x \cos x \cos x + \tan x \sin x \sin x \\ &= \tan x \\ u_1 &= \int \tan x = - \ln \cos x \end{aligned}$$

The general solution:

$$\begin{aligned} y = y_H + y_P =& c_1 + c_2 \cos x + c_3 \sin x \\ & - \ln \cos x \\ & + (\cos x) \cos x \\ & + (\sin x + \ln(\sec x + \tan x)) \sin x \\ =& c_1' + c_2 \cos x + c_3 \sin x \\ & - \ln \cos x - \sin x \ln(\sec x + \tan x)) \end{aligned}$$

Note that we can use the formula in the book for solving the linear system of 3 equations above. But it takes more time to compute 4 determinants that way.

13. Solve an Euler equation:
23.41

$$x^2 y'' + 3xy' - 4y = 0$$

- **ans:**

$$r(r-1) + 3r - 4 = 0$$

$$r = -1 \pm \sqrt{5}$$

$$y = c_1 x^{-1+\sqrt{5}}) + c_2 x^{-1-\sqrt{5}}$$

14. Solve a non homogeneous Euler equation:
23.43

$$2x^2 y'' + 5xy' + y = x^2 - x$$

- **ans:** For $y_H = y_c$:

$$2r(r-1) + 5r + 1 = 0$$

$$r = -1, -\frac{1}{2}$$

$$y = c_1 \frac{1}{x} + c_2 \frac{1}{\sqrt{x}}$$

For y_p, we have to use the method of variation of parameter as we have a non-constant coefficient equation, the method of undeteminend coefficients does not work.

However, we have to normalize the equation first!

$$y'' + \frac{5}{2x} y' + \frac{1}{2x^2} y = \frac{1}{2} - \frac{1}{2x}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$\begin{aligned} u_1' x^{-1} + u_2' x^{-1/2} &= 0 \\ -u_1' x^{-2} - \frac{1}{2} u_2' x^{-3/2} &= \frac{1}{2} - \frac{1}{2x} \end{aligned}$$

$(1/2) eq1/x + eq2$:

$$\begin{aligned} u_1' (\frac{1}{2} - 1) x^{-2} &= \frac{1}{2} - \frac{1}{2x} \\ u_1' &= (-1 + \frac{1}{x}) x^2 \\ u_1 &= -\frac{1}{3} x^3 + \frac{1}{2} x^2 \end{aligned}$$

$eq1/x + eq2$:

$$u_2'(1-\frac{1}{2})x^{-3/2} = \frac{1}{2} - \frac{1}{2x}$$
$$u_2' = (1-\frac{1}{x})x^{3/2}$$
$$u_2 = \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}$$

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 \\ &= (-\frac{1}{3}x^2 + \frac{1}{2}x) + (\frac{2}{5}x^2 - \frac{2}{3}x) \\ &= \frac{1}{15}x^2 - \frac{1}{6}x \end{aligned}$$

The general solution is

$$y = y_H + y_p = c_1\frac{1}{x} + c_2\frac{1}{\sqrt{x}} + \frac{1}{15}x^2 - \frac{1}{6}x$$

15. Taylor series solution of IVP:

823.18

$$y'' + y^2 = 1$$
$$y(0) = 2, \quad y'(0) = 3$$

- **ans:**

$$y'' = -y^2 + 1 \qquad y''(0) = -2^2 + 1 = -3$$

$$y''' = -2yy' \qquad y'''(0) = -12$$

$$y^{(4)} = -2yy'' - 2(y')^2 \qquad y^{(4)}(0) = -6$$

$$y^{(5)} = -2yy''' - 6(y')y'' \qquad y^{(4)}(0) = 102$$

Taylor formula:

$$y = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + ...$$

$$y = 2 + 3x - \frac{3}{2}x^2 - 2x^3 - \frac{1}{4}x^4 + \frac{17}{20}x^5$$

16. Find the charge on the capacitor in an LRC-series circuit when

24.46

$$L = \frac{1}{2}h, \ R = 10\Omega, \ C = 0.01f, \ E = 150V$$

with initial condition

$$q(0) = 1C, \ i(0) = 0A.$$

What is the charge on the capacitor after a long time?

- **ans:**

$$LQ'' + RQ' + C^{-1}Q = E$$
$$\frac{1}{2}Q'' + 10Q' + 100Q = 150$$

Find Q_H:

$$\frac{1}{2}r^2 + 10r + 100 = 0, \ r = -10 \pm 10i$$

$$Q_H = e^{-10t}(A\cos 10t + B\sin 10t)$$

Find Q_P:

$$Q_P = C \ \Rightarrow C = \frac{3}{2}$$

$$Q = Q_H + Q_P = e^{-10t}(A\cos 10t + B\sin 10t) + \frac{3}{2}$$

By

$$Q(0) = 1, \ Q'(0) = 0$$

$$Q(t) = -\frac{1}{2}e^{-10t}(\cos 10t + \sin 10t) + \frac{3}{2}$$

When $t \to \infty$, $e^{-10t} \to 0$

$$Q(t) \to \frac{3}{2}C$$

17. Given three vectors:

27.6

$$\mathbf{u}_1 = \langle 0,1,0\rangle, \ \mathbf{u}_2 = \langle 1,2,0\rangle, \ \mathbf{u}_3 = \langle 1,1,3\rangle,$$

(a) Show linearly independence

(b) Find a linear combination for $\mathbf{a} = \langle 0,1,0\rangle$, i.e, coordinates of $\mathbf{a}$, under the basis, $\mathbf{u}_1, ...$

(c) Find the orthgonal bases by the Gram-Schmidt orthogonalization process, $\mathbf{v}_1$ ....

(d) Find the orthnormal bases by the Gram-Schmidt orthogonalization process, $\mathbf{w}_1$ ...

(e) Find a linear combination for $\mathbf{a} = \langle 0,1,0\rangle$, i.e, coordinates of $\mathbf{a}$, under the orthnormal basis above $\mathbf{w}_1$ ....

- **ans:**

(a) Only zero solutions:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$$

$$\begin{array}{cccc} & c_2 & +c_3 & = 0 \\ c_1 & +2c_2 & +c_3 & = 0 \\ & & 3c_3 & = 0 \end{array}$$

$c_1 = c_2 = c_3 = 0.$

(b) Solve linear system:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{a}$$

$$\begin{array}{rrrl} & c_2 & +c_3 & = 0 \\ c_1 & +2c_2 & +c_3 & = 1 \\ & & 3c_3 & = 0 \end{array}$$

to get

$$c_1 = 1,\ c_2 = 0,\ c_3 = 0$$

$$\langle 0,1,0\rangle = (1)\mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3$$

Coordinates $(1,0,0)$.

(c) Orthogonal basis.

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1 = \langle 0,1,0\rangle \\ \mathbf{v}_2 &= \mathbf{u}_2 - proj_{\mathbf{v}_1}\mathbf{u}_1 = \langle 1,0,0\rangle \\ \mathbf{v}_3 &= \mathbf{u}_3 - proj_{\mathbf{v}_1}\mathbf{u}_3 - proj_{\mathbf{v}_2}\mathbf{u}_3 = \langle 0,0,3\rangle \end{aligned}$$

(d) Orthonormal basis.

$$\mathbf{w}_i = \mathbf{v}_i/\|\mathbf{v}_i\|$$

$$\begin{aligned} \mathbf{w}_1 &= \langle 0,1,0\rangle \\ \mathbf{w}_2 &= \langle 1,0,0\rangle \\ \mathbf{w}_3 &= \langle 0,0,1\rangle \end{aligned}$$

(e) The coordinates under $\mathbf{w}_i$.
2 method:
(1) solving lienar system as in (b):

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{a}$$

$$\begin{array}{rrrl} & c_2 & & = 0 \\ c_1 & & & = 1 \\ & & c_3 & = 0 \end{array}$$

to get

$$c_1 = 1,\ c_2 = 0,\ c_3 = 0$$

$$\langle 0,1,0\rangle = (1)\mathbf{w}_1 + 0\mathbf{w}_2 + 0\mathbf{w}_3$$

Coordinates $(1,0,0)$.
(2) using inner products.

$$c_1 = \mathbf{a}\cdot\mathbf{w}_1 = 1$$

$$c_2 = \mathbf{a}\cdot\mathbf{w}_2 = 0$$

$$c_3 = \mathbf{a}\cdot\mathbf{w}_3 = 0$$

$$\mathbf{a} = 1\mathbf{w}_1 + 0\mathbf{w}_2 + 0\mathbf{w}_3$$

18. (1) Determine if $(A|b)$ is in row-echelon form, reduce it to
303.1303 a row-echelon form if not, and use its row-echelon form to solve $Ax = b$.
(2) Determine if $(A|b)$ is in reduced row-echelon form, reduce it to THE reduced row-echelon form if not, and use THE row-echelon form to solve $Ax = b$.

$$\left(\begin{array}{ccccc|c} 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -2 & 2 & -2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 2 \end{array}\right)$$

• **ans:** (1) No, not in row-echelon form, there are zeros above the first non-zero.

$r_1 \leftrightarrow r_3$:

$$\left(\begin{array}{ccccc|c} 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & -2 & 2 & -2 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \end{array}\right)$$

$(-1/2)r_2$:

$$\left(\begin{array}{ccccc|c} 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \end{array}\right)$$

$-r_2 + r_3$: (note that it is illegal to write $r_3 - r_2$!)

$$\left(\begin{array}{ccccc|c} 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We get REF. The rank is 2. We have 5 columns for A, so we have $5-2$, three free variables, x_1, x_4 and x_5:

$$x = \begin{pmatrix} c_1 \\ * \\ * \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ 2 - c_2 + c_3 \\ c_2 - c_3 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= x_H + x_p = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(2) No, not in reduced row-echelon form.

By above row operations, we end with the RREF too:

$$\left(\begin{array}{ccccc|c} 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We get the same solution.

19. Find the linearly dependence by rank and the number of
304.1103 free variables for the homogeneous system $Ax = 0$.

$$(1)\mathbf{u}_i = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad (2)\mathbf{u}_i = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}.$$

• **ans:** (1) We write vectors columnwise to form a matrix, then find the rank:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{pmatrix}$$
$$\overset{-r_1+r_3}{\longrightarrow} \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ & -2 \end{pmatrix}$$

rank is 2 (this is not REF, but enough to tell the nonzero rows in the REF). Since we have two columns in A, we have $2 - 2 = 0$ free variable in $Ax = 0$. So, there is no non zero solution. They are linearly independent.

(2) We write vectors columnwise to form a matrix, then find the rank:

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \\ 1 & -1 & 3 \end{pmatrix}$$
$$\overset{-r_1+r_3}{\longrightarrow} \begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \\ & -2 & -6 \end{pmatrix}$$

The rank is 2 (this is not REF, but enough to tell the nonzero rows in the REF). Since we have three columns in A, we have $3-2 = 1$ free variable in $Ax = 0$. In this case, x_3 can be free. So, there are non zero solutions (for example, let $x_3 = 1$.) They are linearly dependent.

20. Find the determinant
306.1206 (1) by cofactor expansions (no row/column operations)
(2) by row operations to an upper triangular matrix.

$$A = \begin{pmatrix} 2 & 0 & 4 & 4 \\ 0 & -10 & 10 & 10 \\ 0 & 0 & 4 & 4 \\ 2 & 2 & 0 & 2 \end{pmatrix}$$

• **ans:** (1)

$$\begin{aligned}
|A| &\overset{\text{by } r_3}{=} 0 - 0 + 4 \begin{vmatrix} 2 & 0 & 4 \\ 0 & -10 & 10 \\ 2 & 2 & 2 \end{vmatrix} - 4 \begin{vmatrix} 2 & 0 & 4 \\ 0 & -10 & 10 \\ 2 & 2 & 0 \end{vmatrix} \\
&\overset{\text{by } r_1, r1}{=} 4(2 \begin{vmatrix} -10 & 10 \\ 2 & 2 \end{vmatrix} - 0 + 4 \begin{vmatrix} 0 & -10 \\ 2 & 2 \end{vmatrix}) \\
&\quad - 4(2 \begin{vmatrix} -10 & 10 \\ 2 & 0 \end{vmatrix} - 0 + 4 \begin{vmatrix} 0 & -10 \\ 2 & 2 \end{vmatrix}) \\
&= 4(2(-20-20) + 4(0+20)) \\
&\quad - 4(2(0-20) + 4(0+20)) \\
&= 0 - 4(40) = -160
\end{aligned}$$

(2) Factor out 2, -10, 4 and 2 from rows:

$$\begin{aligned}
|A| &\overset{\frac{1}{2}r_1, \frac{1}{-10}r_2, \frac{1}{4}r_3, \frac{1}{2}r_4}{=} (2)(-10)(4)(2) \begin{vmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix} \\
&\overset{-r_1+r_4}{=} -160 \begin{vmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ & 1 & -2 & -1 \end{vmatrix} \\
&\overset{-r_2+r_4}{=} -160 \begin{vmatrix} 1 & 0 & 2 & 2 \\ & 1 & -1 & -1 \\ & & 1 & 1 \\ & 0 & -1 & 0 \end{vmatrix} \\
&\overset{r_3+r_4}{=} (-160) \begin{vmatrix} 1 & 0 & 2 & 2 \\ & 1 & -1 & -1 \\ & & 1 & 1 \\ & & & 1 \end{vmatrix} \\
&= -160(1)(1)(1)(1) = -160
\end{aligned}$$

21. Find A^{-1} by (1) cofactors, (2) row operations. And use
307.1303 A^{-1} to solve $Ax = b$.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

• **ans:** (1) Cofactor method:

$$\begin{aligned}
A^{-1} &= \frac{1}{|A|} C^T \\
&= \frac{1}{|A|} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}^T \\
&= \frac{1}{\begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{vmatrix}} \begin{pmatrix} \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} \end{pmatrix}^T \\
&= \frac{1}{1} \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 3 & -1 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}
\end{aligned}$$

(2) Row operation method:

$$
\begin{aligned}
(A\ I) &= \begin{pmatrix}1 & 1 & 2\\ 0 & -1 & 1\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & & \\ 0 & 1 & \\ & & 1\end{pmatrix}\\
&\overset{-r2,-r_3}{\longrightarrow}\begin{pmatrix}1 & 1 & 2\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & & \\ 0 & -1 & \\ & & -1\end{pmatrix}\\
&\overset{-r_2+r_1}{\longrightarrow}\begin{pmatrix}1 & 0 & 3\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & \\ 0 & -1 & \\ & & -1\end{pmatrix}\\
&\overset{-3r_3+r_1}{\longrightarrow}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 3\\ 0 & -1 & \\ & & -1\end{pmatrix}\\
&\overset{r_3+r_2}{\longrightarrow}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 3\\ 0 & -1 & -1\\ & & -1\end{pmatrix}\\
&= (I\ A^{-1})
\end{aligned}
$$

$$A^{-1} = \begin{pmatrix}1 & 1 & 3\\ 0 & -1 & -1\\ & & -1\end{pmatrix}$$

Solution:

$$x = A^{-1}b = \begin{pmatrix}1 & 1 & 3\\ 0 & -1 & -1\\ & & -1\end{pmatrix}\begin{pmatrix}0\\ 0\\ -1\end{pmatrix} = \begin{pmatrix}-3\\ 1\\ 1\end{pmatrix}$$

22. Factor the matrix A into a product XDX^{-1}, where D is
325.400d diagonal. Then find A^{132} by the factorization.

$$A = \begin{pmatrix}1 & 0 & 0\\ -1 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

- **ans:** (a) Find eigenvalues:

$$
\begin{aligned}
\det(A-\lambda I) &= \begin{pmatrix}1-\lambda & 0 & 0\\ -1 & -1-\lambda & 0\\ 0 & 0 & 1-\lambda\end{pmatrix}\\
&= (1-\lambda)(-1-\lambda)(1-\lambda) = 0\\
\lambda &= 1, 1, -1
\end{aligned}
$$

(Do not multiply out the product then factor the result back into a product to find three roots.)

For $\lambda = 1$, solve $(A-\lambda I)x = 0$:

$$\begin{pmatrix}0 & 0 & 0\\ -1 & -2 & 0\\ 0 & 0 & 0\end{pmatrix},\ x = c_1\begin{pmatrix}2\\ -1\\ 0\end{pmatrix} + c_2\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

(note: the rank is 1, three columns, so $3-1 = 2$ free variables, x_2 and x_3.)

For $\lambda = -1$, solve $(A-\lambda I)x = 0$:

$$\begin{pmatrix}2 & 0 & 0\\ -1 & 0 & 0\\ 0 & 0 & 2\end{pmatrix},\ x = c\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$$

(note: the rank is 2, three columns, so $3-2 = 1$ free variable, x_2.)

Write three eigenvectors columnwise to get Z:

$$
\begin{aligned}
X &= \begin{pmatrix}2 & 0 & 0\\ -1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\\
X^{-1} &= \frac{1}{|X|}C^T\\
&= \frac{1}{|X|}\begin{pmatrix}c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\\ c_{31} & c_{32} & c_{33}\end{pmatrix}^T\\
&= \begin{pmatrix}1/2 & 0 & 0\\ 0 & 0 & 1\\ 1/2 & 10 & \end{pmatrix}\\
A &= XDX^{-1}\\
&= X\begin{pmatrix}1 & & \\ & 1 & \\ & & -1\end{pmatrix}X^{-1}\\
A^{132} &= XD^{132}X^{-1}\\
&= X\begin{pmatrix}1^{132} & & \\ & 1^{132} & \\ & & (-1)^{132}\end{pmatrix}X^{-1}\\
&= X\begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix}X^{-1}\\
&= XX^{-1} = I = \begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix}
\end{aligned}
$$

23. Solve the system by (a) elimination, (b) eigenfunctions:
43.27

$$x_1' = 5x_1 - 9x_2,\ x_2' = 2x_1 - x_2$$

- **ans:** Method (by elimination)

By the first equation

$$x_2 = -\frac{1}{9}x_1' + \frac{5}{9}x_1$$

Plug it into the second equation:

$$-\frac{1}{9}x_1'' + \frac{5}{9}x_1' = 2x_1 + \frac{1}{9}x_1' - \frac{5}{9}x_1$$

$$x_1'' - 4x_1' + 13x_1 = 0$$

Solving the homogeneous equation:

$$r^2 - 4r + 13 = 0$$

$$(r-2)^2+3^2=0$$

$$r=2\pm 3i$$

$$x_1=C_1e^{2t}\cos 3t+C_2e^{2t}\sin 3t$$

$$\begin{aligned} x_2 &= -\frac{1}{9}x_1'+\frac{5}{9}x_1 \\ &= \frac{1}{3}e^{2t}(C_2(\sin 3t-\cos 3t)C_1(\cos 3t+\sin 3t) \end{aligned}$$

Method (by eigenvalues)

$$\det(A-\lambda I)=0,\ \lambda=2\pm 3i$$

For $\lambda=2+3i$

$$\begin{pmatrix} 3-3i & -9 & | & 0 \\ 2 & -3-3i & | & 0 \end{pmatrix} \rightarrow x=C\begin{pmatrix} 9 \\ 3-3i \end{pmatrix}$$

$$\begin{aligned} x &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= C_1e^{2t}\left(\cos 3t\begin{pmatrix} 9 \\ 3 \end{pmatrix}-\sin 3t\begin{pmatrix} 0 \\ -3 \end{pmatrix}\right) \\ &\quad +C_2e^{2t}\left(\cos 3t\begin{pmatrix} 0 \\ -3 \end{pmatrix}+\sin 3t\begin{pmatrix} 9 \\ 3 \end{pmatrix}\right) \end{aligned}$$

24. Solve the system by elimination and by eigenfunctions:
327.4

$$x_1'=-x_1-x_2,\ x_2'=4x_1-x_2$$

- **ans:**

$$\det(A-\lambda I)=0,\ \lambda=-1\pm 2i$$

For $\lambda=-1+2i$ (only need an eigenvector for the plus root, no need for the minus root.)

$$\begin{pmatrix} -2i & -1 & | & 0 \\ 4 & -2i & | & 0 \end{pmatrix} \rightarrow \mathbf{v}=C\begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

Separate the eigenvector into real and imaginary parts:

$$\mathbf{v}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}+i\begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

Two linearly independent solutions are (real*real minus imaginary*imaginary) and (real*imaginary plus imaginary*real):

$$\begin{aligned} x &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= C_1e^{-t}\left(\cos 2t\begin{pmatrix} 1 \\ 0 \end{pmatrix}-\sin 2t\begin{pmatrix} 0 \\ -2 \end{pmatrix}\right) \\ &\quad +C_2e^{-t}\left(\cos 2t\begin{pmatrix} 0 \\ -2 \end{pmatrix}+\sin 2t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \end{aligned}$$

Method(by elimination) By the first equation

$$x_2=-x_1'-x_1$$

Plug x_2 into the second equation:

$$\begin{aligned} (-x_1'-x_1)' &= 4x_1-(-x_1'-x_1) \\ x_1''-2x_1'+5x_1 &= 0 \\ r^2-2r+5 &= 0 \\ (r-1)^2 &= -4 \Rightarrow r=-1\pm 2i \\ x_1 &= Ae^{-t}\cos 2t+Be^{-t}\sin 2t \end{aligned}$$

By the equation above for x_2:

$$\begin{aligned} x_2 &= -x_1'-x_1 \\ &= -Ae^{-t}\cos 2t-2Ae^{-t}\sin 2t \\ &\quad -Be^{-t}\sin 2t+2Be^{-t}\cos 2t \end{aligned}$$

25. Solve the system $x'=Ax$ by eigenfunctions:
327.6

$$A=\begin{pmatrix} -3 & 1 & -1 \\ 0 & -2 & 0 \\ 2 & 2 & -1 \end{pmatrix}$$

- **ans:**

$$\det(A-\lambda I)=0,\ \lambda=-2,-2\pm i$$

For $\lambda=-2$

$$(A-\lambda I)x=0 \rightarrow x=C\begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$$

For $\lambda=-2+i$ (only for the plus eigenvalue)

$$(A-\lambda I)x=0 \rightarrow x=C\begin{pmatrix} 1 \\ 0 \\ 1+i \end{pmatrix}$$

Separate the eigenvector into real and imaginary parts:

$$\mathbf{v}=\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}+i\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Two linearly independent solutions are (real*real minus imaginary*imaginary) and (real*imaginary plus imaginary*real):

$$\begin{aligned} x &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= C_1e^{-2t}\begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} \\ &\quad +C_2e^{-2t}\left(\cos t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}-\sin t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \\ &\quad +C_3e^{-2t}\left(\cos t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}+\sin t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right) \end{aligned}$$

26. Find the general solution of the homogeneous system by
47.6 the method of eigenvalue:

$$x' = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} x$$

- **ans:** $r = 2, 2, 2$

$$r = 2, \begin{pmatrix} 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \to v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

NOT enough v for $r = 2$! We need put one of them back to find v_3. Which one do we use, v_1 or v_2? Put v_1 back (if we use v_2, there is no solution!)

$$\begin{pmatrix} 0 & 0 & 0 & | & 1 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \to v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$x = C_1 e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_3 e^{2t} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

27. Solve the system $x' = Ax$ by eigenfunctions (repeated eigen-
327.8 values), if $A =$

$$(a) \begin{pmatrix} 2 & 1 & 6 \\ & 2 & 5 \\ & & 3 \end{pmatrix} \quad (b) \begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & -1 \end{pmatrix} \quad (c) \begin{pmatrix} 1 & 1 & 1 \\ & 1 & 0 \\ & & -1 \end{pmatrix}$$

- **ans:** (a) Find eigenvalues and eigenvectors:

$$\det(A - \lambda I) = (2-\lambda)(2-\lambda)(3-\lambda) = 0,$$
$$\lambda = 2, 2, 3$$

(note: the matrix is triangular, also do not multiply out the product, then factor the result back.)

For $\lambda = 2, 2$, solve $(A - \lambda I)x = 0$,

$$(A - \lambda I) = \begin{pmatrix} 0 & 1 & 6 \\ & & 5 \\ & & 1 \end{pmatrix}$$

Rank 2, three columns, $3 - 2 = 1$ free variable, x_1.

$$k_1 = C \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We need two vectors. So we put the solution back to solve $(A - \lambda I)x = k_1$,

$$\begin{pmatrix} 0 & 1 & 6 & | & 1 \\ & & 5 & | & 0 \\ & & 1 & | & 0 \end{pmatrix}$$

Rank 2, three columns, $3 - 2 = 1$ free variable, x_1. But to find one vector only, we let $x_1 = 0$

$$k_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda = 3$

$$(A - \lambda I) = \begin{pmatrix} -1 & 1 & 6 \\ & -1 & 5 \\ & & 0 \end{pmatrix}$$

Rank 2, three columns, $3 - 2 = 1$ free variable, x_3.

$$k_3 = C \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So we can put three vectors together to find $x = x_H$:

$$x = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} [\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}] + c_3 e^{3t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(b) Find eigenvalues and eigenvectors:

$$\det(A - \lambda I) = (1-\lambda)(1-\lambda)(-1-\lambda) = 0,$$
$$\lambda = 1, 1, -1$$

(note: the matrix is triangular, also do not multiply out the product, then factor the result back.)

For $\lambda = 1, 1$, solve $(A - \lambda I)x = 0$,

$$(A - \lambda I) = \begin{pmatrix} 0 & 0 & 1 \\ & & 0 \\ & & -2 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 1 \\ & & 0 \\ & & 0 \end{pmatrix}$$

Rank 1, three columns, $3 - 1 = 2$ free variable, x_1, x_2.

$$k_1 = C \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

We need two vectors and we have two here. No need to find more vectors.

For $\lambda = -1$

$$(A - \lambda I) = \begin{pmatrix} 2 & 0 & 1 \\ & 2 & 0 \\ & & 0 \end{pmatrix}$$

Rank 2, three columns, $3 - 2 = 1$ free variable, x_3.

$$k_3 = C \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

So we can put three vectors together to get $x = x_H$:

$$x = c_1 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

(c) Find eigenvalues and eigenvectors:

$$\det(A-\lambda I)=(1-\lambda)(1-\lambda)(-1-\lambda)=0,$$
$$\lambda=1,1,-1$$

(note: the matrix is triangular, also do not multiply out the product, then factor the result back.)

For $\lambda=1,1$, solve $(A-\lambda I)x=0$,

$$(A-\lambda I)=\begin{pmatrix}0&1&1\\ &0&0\\ & &-2\end{pmatrix}$$

Rank 2, three columns, $3-2=1$ free variable, x_1.

$$k_1=C\begin{pmatrix}1\\0\\0\end{pmatrix}$$

We need two vectors. So we put the solution back to solve $(A-\lambda I)x=k_1$,

$$\left(\begin{array}{ccc|c}0&1&1&1\\ & &0&0\\ & &-2&0\end{array}\right)$$

Rank 2, three columns, $3-2=1$ free variable, x_1. But to find one vector only, we let $x_1=0$

$$k_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

For $\lambda=-1$

$$(A-\lambda I)=\begin{pmatrix}2&1&1\\ &2&0\\ & &0\end{pmatrix}$$

Rank 2, three columns, $3-2=1$ free variable, x_3.

$$k_3=C\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

So we can put three vectors together to find $x=x_H$:

$$x=c_1e^t\begin{pmatrix}1\\0\\0\end{pmatrix}+c_2e^t[\begin{pmatrix}1\\0\\0\end{pmatrix}t+\begin{pmatrix}0\\1\\0\end{pmatrix}]+c_3e^{-t}\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

28. Solve the system $x'=Ax+g$ by the eigenfunctions and the
335.11 undetermined coefficients:

$$A=\begin{pmatrix}2&-1\\3&-2\end{pmatrix},\ g=\begin{pmatrix}0\\t\end{pmatrix}$$

- **ans:**

$$\det(A-rI)=0,\ r=-1,1$$

For $r=-1$

$$(A-rI)x=0\ \rightarrow\ x=C\begin{pmatrix}1\\3\end{pmatrix}$$

For $r=1$

$$(A-rI)x=0\ \rightarrow\ x=C\begin{pmatrix}1\\1\end{pmatrix}$$

$$x_H=c_1e^{-t}\begin{pmatrix}1\\3\end{pmatrix}+c_2e^t\begin{pmatrix}1\\1\end{pmatrix}$$

For

$$g=\begin{pmatrix}0\\t\end{pmatrix}$$

we let

$$x_p=\begin{pmatrix}A_1\\A_2\end{pmatrix}t+\begin{pmatrix}B_1\\B_2\end{pmatrix}$$

By the differential equation

$$\begin{aligned}&\begin{pmatrix}A_1\\A_2\end{pmatrix}\\ &=\begin{pmatrix}2A_1-A_2\\3A_1-2A_2\end{pmatrix}t+\begin{pmatrix}2B_1-B_2\\3B_1-2B_2\end{pmatrix}+\begin{pmatrix}0\\t\end{pmatrix}\end{aligned}$$

$$\begin{aligned}A_1&=2B_1-B_2\\A_2&=3B_1-2B_2\\2A_1-A_2&=0\\3A_1-2A_2+1&=0\end{aligned}$$

$$\begin{aligned}A_1&=1\\A_2&=2\\B_1&=0\\B_2&=-1\end{aligned}$$

$$x_p=\begin{pmatrix}t\\2t-1\end{pmatrix}$$

$$\begin{aligned}x&=x_H+x_p\\&=c_1e^{-t}\begin{pmatrix}1\\3\end{pmatrix}+c_2e^t\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}t\\2t-1\end{pmatrix}\end{aligned}$$

```
eq:=diff(x(t),t)=2x(t)-y(t)+exp(t),
 diff(y(t),t)=3x(t)-2y(t)+t;
dsolve({eq},
  {x(t),y(t)});
dsolve({eq,x(0)=1/2,y(0)=-1},
  {x(t),y(t)});
```

29. Solve the system $x' = Ax + g$ by the eigenfunctions and the
335.14 undetermined coefficients:

$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}, \; g = \begin{pmatrix} 4e^t \\ 2e^t \end{pmatrix}$$

- **ans:**

$$\det(A - rI) = 0, \; (r^2 - 2^2) + 5 = 0$$

$$r = \pm i$$

For $r = i$ (no need to find eigenvectors for the minus sign eigenvalue)

$$(A - rI)x = 0 \; \rightarrow \; \begin{pmatrix} 2-i & -5 \\ 0 & 0 \end{pmatrix}$$

$$x = C\begin{pmatrix} 5 \\ 2-i \end{pmatrix} = C\begin{pmatrix} 5 \\ 2 \end{pmatrix} + Ci\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Two linearly independent solutions are (real*real minus imaginary*imaginary) and (real*imaginary plus imaginary*real):

$$\begin{aligned} x_H =& c_1(\cos t\begin{pmatrix} 5 \\ 2 \end{pmatrix} - \sin t\begin{pmatrix} 0 \\ -1 \end{pmatrix}) \\ &+ c_2(\cos t\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \sin t\begin{pmatrix} 5 \\ 2 \end{pmatrix}) \end{aligned}$$

For

$$g = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^t$$

we set up

$$x_p = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^t$$

By the differential equation

$$\begin{aligned} &\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \\ =& \begin{pmatrix} 2A_1 - 5A_2 \\ A_1 - 2A_2 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A_1 &= 2A_1 - 5A_2 + 4 \\ A_2 &= A_1 - 2A_2 + 2 \end{aligned}$$

$$A_1 = 1$$

$$A_2 = 1$$

$$x_p = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$$

$$x = x_H + x_p$$

30. Solve the system $x' = Ax + F$ by the eigenfunctions and
335.1 (a) the undetermined coefficients,
(b) the method of Variation of parameters,
(c) the method of Diagonalization,
(d) by direct elimination.

$$A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}, \; F = \begin{pmatrix} 0 \\ -8 \end{pmatrix} e^t$$

- **ans:** First, we find x_H for the system $x' = Ax$.

$$\det(A - rI) = 0, \; r = 2, -3$$

For $r = 2$

$$(A - rI)x = 0 \; \rightarrow \; \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$

Rank, 2 columns, one free variable, x_2:

$$x = C\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $r = -3$

$$(A - rI)x = 0 \; \rightarrow \; \begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}$$

Rank, 2 columns, one free variable, x_2:

$$x = C\begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$$x_H = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

(a) Note: 1 is not a root:
For

$$F = \begin{pmatrix} 0 \\ -8 \end{pmatrix} e^t$$

we let

$$x_p = ce^t = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^t$$

By the differential equation (replace x there by x_p):

$$\begin{aligned} &\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \\ =& \begin{pmatrix} A_1 + A_2 \\ 4A_1 - 2A_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -8 \end{pmatrix} \end{aligned}$$

Two equations and two unknowns, find them:

$$x_p = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^t$$

$$x = x_H + x_p$$

(b) by x_H:

$$\begin{aligned}
\Phi &= \begin{pmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{pmatrix} \\
\Phi^{-1} &= \frac{1}{\begin{vmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{vmatrix}} \begin{pmatrix} -4e^{-3t} & -e^{2t} \\ -e^{-3t} & e^{2t} \end{pmatrix}^T \\
&= -\frac{1}{5} e^t \begin{pmatrix} -4e^{-3t} & -e^{-3t} \\ -e^{2t} & e^{2t} \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} 4e^{-2t} & e^{-2t} \\ e^{3t} & -e^{3t} \end{pmatrix} \\
x_P &= \Phi(t) \int \Phi^{-1} F dt \\
&= \Phi(t) \int \frac{1}{5} \begin{pmatrix} 4e^{-2t} & e^{-2t} \\ e^{3t} & -e^{3t} \end{pmatrix} \begin{pmatrix} 0 \\ -8 \end{pmatrix} e^t dt \\
&= \Phi(t) \int \frac{8}{5} \begin{pmatrix} -e^{-t} \\ e^{4t} \end{pmatrix} dt \\
&= \begin{pmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{pmatrix} \begin{pmatrix} (8/5)e^{-t} \\ (2/5)e^{4t} \end{pmatrix} \\
&= \begin{pmatrix} 10/5 \\ 0 \end{pmatrix} e^t
\end{aligned}$$

$$x = x_H + x_p$$

(c) By (a)

$$\begin{aligned}
P &= \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}, \; P^{-1} = -\frac{1}{5} \begin{pmatrix} -4 & -1 \\ -1 & 1 \end{pmatrix}^T = \frac{1}{5} \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \\
x &= Py \\
y' &= Dy + P^{-1}F \\
&= \begin{pmatrix} 2 & \\ & -3 \end{pmatrix} y + \frac{1}{5} \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -8 \end{pmatrix} e^t \\
&= \begin{pmatrix} 2y_1 - (8/5)e^t \\ -3y_2 + (8/5)e^t \end{pmatrix}
\end{aligned}$$

We can write the system from matrix form to separated equations:

$$\begin{aligned}
y_1' &= 2y_1 - (8/5)e^t \\
y_2' &= -3y_2 + (8/5)e^t
\end{aligned}$$

We solve the two equations separately:

$$\begin{aligned}
y_1' &= 2y_1 - (8/5)e^t \\
y_{1H} &= c_1 e^{2t} \\
y_{1p} &= Ae^t, \; y_{1p} = (8/5)e^t \\
y_1 &= y_{1H} + y_{1p} \\
y_2' &= -3y_2 + (8/5)e^t \\
y_{2H} &= c_2 e^{-3t} \\
y_{2p} &= Ae^t, \; y_{2p} = (2/5)e^t \\
y_2 &= y_{2H} + y_{2p}
\end{aligned}$$

We then write them together in a vector form:

$$\begin{aligned}
y &= \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} + (8/5)e^t \\ c_2 e^{-3t} + (2/5)e^t \end{pmatrix} \\
x &= Py = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} c_1 e^{2t} + (8/5)e^t \\ c_2 e^{-3t} + (2/5)e^t \end{pmatrix} \\
&= \begin{pmatrix} c_1 e^{2t} + c_2 e^{-3t} + 2e^t \\ c_1 e^{2t} - 4c_2 e^{-3t} \end{pmatrix}
\end{aligned}$$

(d) Elimination: (this is like the method (c) above, but without finding x_H at all.)

We write the original system in equation form:

$$\begin{aligned}
x_1' &= x_1 + x_2 \\
x_2' &= 4x_1 - 2x_2 - 8e^t
\end{aligned}$$

By the first equation, we get

$$x_2 = x_1' - x_1$$

So we replace x_2 in the equation 2 by this relation:

$$\begin{aligned}
x_1'' - x_1' &= 4x_1 - 2x_1' + 2x_1 - 8e^t \\
x_1'' + x_1' - 6x_1 &= -8e^t
\end{aligned}$$

We find x_{1H} and x_{1p} as follows:

$$\begin{aligned}
r^2 + r - 6 = 0, \; r &= 2, -3 \\
x_{1p} &= Ae^t, \; x_{1p} = 2e^t \\
x_1 &= x_{1H} + x_{1p} = c_1 e^{2t} + c_2 e^{-3t} + 2e^t
\end{aligned}$$

Again, by the above equation (obtained from equation 1):

$$x_2 = x_1' - x_1 = c_1 e^{2t} - 4c_2 e^{-3t}$$