

M351 Study Guide Final (S. Zhang) .

1. Solve

$$y' + y = \begin{cases} 1 & x < 1 \\ -1 & x \geq 1 \end{cases} \quad y(0) = 1$$

2. Construct a table for y' , y'' and solution curve $y(x)$ shapes. Find and classify critical points Sketch phase portrait (phase line + direction field). Sketch all typical solution curves on the graph of direction field.

$$y' = 2y - y^2.$$

3. Solve the IVP. Determine an interval for which the initial value problem is certain to have a unique solution.

$$y' = \frac{2t}{2 - 2y}, \quad y(3) = 2.$$

4. Determine if the differential equation is exact, and solve it if it is.

$$(2y - \frac{1}{x} + \cos 3x) \frac{dy}{dx} + \frac{y}{x^2} - 4x^3 + 3y \sin 3x = 0$$

5. Solve the equation of $y' = f(y/x)$ form:

$$\frac{dy}{dx} = \frac{x + 3y}{3x + y}$$

6. A tank contains 100 liters of salt water with a concentration of 2 g/liter. Pure water flows in at a rate of 2 liters/min, the well-stirred solution flowing out at the same rate. Find the time when the tank reaches 1 g/liter salt concentration.

7. The time rate of change of an alligator population P in a swamp is proportional to the square of P . The swamp contained a dozen alligators in 1988, two dozen in 1998. When will there be four dozen alligators in the swamp? What happens there after?

8. Find the linear dependence by both finding nonzero solutions and by Wronskian:

$$y_1 = 1 + x^2, \quad y_2 = x^2 - x, \quad y_3 = x^2 + x$$

9. Find the roots of characteristic equation and the general solution:

$$(1) 4y'' - 4y' + y = 0$$

$$(2) (D - 1)^2(D^2 - 1)^2((D - 2)^2 - 9)((D - 1)^2 + 4)^2y = 0$$

10. Solve

$$y'' - 2y' - 3y = 3e^{2t}.$$

11. Find the general solution of the homogeneous equation y_c and write down the form for undetermined coefficients for y_p . (Do not solve for y_p .)

$$(D^2 - 9)(D + 3)(D^2 + 4)y = e^x \sin x - e^{-3x} + \cos 2x.$$

12. Solve

$$y''' + y' = \tan x$$

13. Solve an Euler equation:

$$x^2y'' + 3xy' - 4y = 0$$

14. Solve a non homogeneous Euler equation:

$$2x^2y'' + 5xy' + y = x^2 - x$$

15. Taylor series solution of IVP:

$$y'' + y^2 = 1$$

$$y(0) = 2, \quad y'(0) = 3$$

16. Find the charge on the capacitor in an LRC-series circuit when

$$L = \frac{1}{2}h, \quad R = 10\Omega, \quad C = 0.01f, \quad E = 150V$$

with initial condition

$$q(0) = 1C, \quad i(0) = 0A.$$

What is the charge on the capacitor after a long time?

17. Given three vectors:

$$\mathbf{u}_1 = \langle 0, 1, 0 \rangle, \quad \mathbf{u}_2 = \langle 1, 2, 0 \rangle, \quad \mathbf{u}_3 = \langle 1, 1, 3 \rangle,$$

(a) Show linearly independence

(b) Find a linear combination for $\mathbf{a} = \langle 0, 1, 0 \rangle$, i.e, coordinates of \mathbf{a} , under the basis, \mathbf{u}_1, \dots

(c) Find the orthogonal bases by the Gram-Schmidt orthogonalization process, $\mathbf{v}_1 \dots$

(d) Find the orthonormal bases by the Gram-Schmidt orthogonalization process, $\mathbf{w}_1 \dots$

(e) Find a linear combination for $\mathbf{a} = \langle 0, 1, 0 \rangle$, i.e, coordinates of \mathbf{a} , under the orthonormal basis above $\mathbf{w}_1 \dots$

18. (1) Determine if $(A|b)$ is in row-echelon form, reduce it to a row-echelon form if not, and use its row-echelon form to solve $Ax = b$.

(2) Determine if $(A|b)$ is in reduced row-echelon form, reduce it to THE reduced row-echelon form if not, and use THE row-echelon form to solve $Ax = b$.

$$\left(\begin{array}{cccc|c} 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -2 & 2 & -2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 2 \end{array} \right)$$

19. Find the linearly dependence by rank and the number of free variables for the homogeneous system $Ax = 0$.

$$(1)\mathbf{u}_i = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad (2)\mathbf{u}_i = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}.$$

20. Find the determinant

- (1) by cofactor expansions (no row/column operations)
(2) by row operations to an upper triangular matrix.

$$A = \begin{pmatrix} 2 & 0 & 4 & 4 \\ 0 & -10 & 10 & 10 \\ 0 & 0 & 4 & 4 \\ 2 & 2 & 0 & 2 \end{pmatrix}$$

21. Find A^{-1} by (1) cofactors, (2) row operations. And use A^{-1} to solve $Ax = b$.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

22. Factor the matrix A into a product DX^{-1} , where D is diagonal. Then find A^{132} by the factorization.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

23. Solve the system by (a) elimination, (b) eigenfunctions:

$$x'_1 = 5x_1 - 9x_2, \quad x'_2 = 2x_1 - x_2$$

24. Solve the system by elimination and by eigenfunctions:

$$x'_1 = -x_1 - x_2, \quad x'_2 = 4x_1 - x_2$$

25. Solve the system $x' = Ax$ by eigenfunctions:

$$A = \begin{pmatrix} -3 & 1 & -1 \\ 0 & -2 & 0 \\ 2 & 2 & -1 \end{pmatrix}$$

26. Find the general solution of the homogeneous system by the method of eigenvalue:

$$x' = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} x$$

27. Solve the system $x' = Ax$ by eigenfunctions (repeated eigenvalues), if $A =$

$$(a) \begin{pmatrix} 2 & 1 & 6 \\ 2 & 5 & \\ 3 & & \end{pmatrix} \quad (b) \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & \\ -1 & & \end{pmatrix} \quad (c) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & & \end{pmatrix}$$

28. Solve the system $x' = Ax + g$ by the eigenfunctions and the undetermined coefficients:

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ t \end{pmatrix}$$

29. Solve the system $x' = Ax + g$ by the eigenfunctions and the undetermined coefficients:

$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}, \quad g = \begin{pmatrix} 4e^t \\ 2e^t \end{pmatrix}$$

30. Solve the system $x' = Ax + F$ by the eigenfunctions and

- (a) the undetermined coefficients,
(b) the method of Variation of parameters,
(c) the method of Diagonalization,
(d) by direct elimination.

$$A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ -8 \end{pmatrix} e^t$$