M351 Study Guide 3 (S. Zhang) .

1. Find counter-examples:

$$AB \neq BA, \ (A+B)(A-B) \neq A^2 - B^2$$

- **ans:**

$$A = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$$

2. Find $B^T(2A)$ and verify $A^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$, if

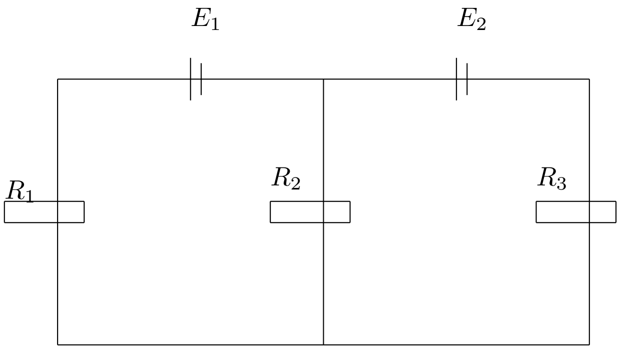
$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \ B = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

- **ans:**

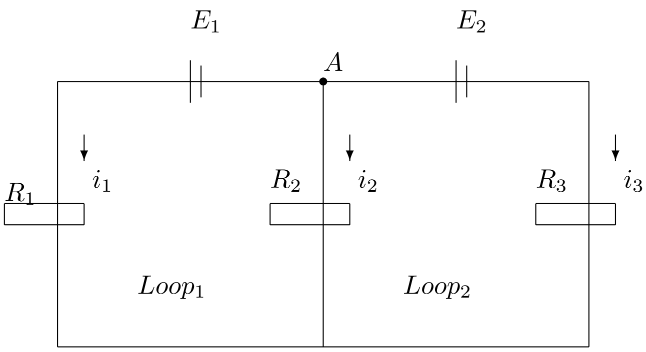
$$B^T(2A) = (2B^T)A = \begin{pmatrix} -2 & 4 \end{pmatrix} A = \begin{pmatrix} 8 & 6 \end{pmatrix}$$

Compute $A^{-1}A = AA^{-1} = I$.

3. Currents in a network. $R_1 = 3$ohms, $R_2 = 5$, $R_3 = 6$, $E_1 = 10$volts, $E_2 = 27$volts, . Find currents.



- **ans:** Define unkowns i_1, i_2, i_3,



Point rule: sum of currents is 0

Loop rule: sum of voltage drops is 0

$$\begin{aligned} -i_1 - i_2 - i_3 &= 0 && \text{(At point } A) \\ R_1 i_1 - R_2 i_2 &= E_1 && (\ Loop_1) \\ R_2 i_2 - R_3 i_3 &= E_2 && (\ Loop_2) \end{aligned}$$

We reduce the matrix form $(A|b)$ of the system to its rref:

$$\left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 3 & -5 & 0 & 10 \\ & 5 & -6 & 27 \end{array}\right) \overset{3r_1+r_2}{\longrightarrow} \left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ & -8 & -3 & 10 \\ & 5 & -6 & 27 \end{array}\right)$$

$$\overset{(-1)r_1}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ & -8 & -3 & 10 \\ & 5 & -6 & 27 \end{array}\right)$$

$$\overset{(-1/8)r_2}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ & 1 & 3/8 & -5/4 \\ & 5 & -6 & 27 \end{array}\right)$$

$$\overset{(-5)r_2+r_3}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ & 1 & 3/8 & -5/4 \\ & & -63/8 & 133/4 \end{array}\right)$$

$$\overset{(-8/63)r_2+r_3}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ & 1 & 3/8 & -5/4 \\ & & 1 & -38/9 \end{array}\right)$$

$$\overset{(-3/8)r_3+r_2}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ & 1 & & 1/3 \\ & & 1 & -38/9 \end{array}\right)$$

$$\overset{(-1)r_3+r_1}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & & 38/9 \\ & 1 & & 1/3 \\ & & 1 & -38/9 \end{array}\right)$$

$$\overset{(-1)r_2+r_1}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & & & 35/9 \\ & 1 & & 1/3 \\ & & 1 & -38/9 \end{array}\right)$$

$$x = \begin{pmatrix} 35/9 \\ 1/3 \\ -38/9 \end{pmatrix}$$

4. Solve the linear system by
(1) Gaussian elimination (must reach Row-echelon Form)
(2) Gauss-Jordan elimination (reach Reduced Row-echelon Form)

$$\begin{array}{rrrr} x_1 & +2x_2 & -4x_3 = & 9 \\ 5x_1 & -x_2 & +2x_3 = & 1 \end{array}$$

- **ans:** (1) Row-echelon form:

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & 9 \\ 5 & -1 & 2 & 1 \end{array}\right) \overset{(-5)r_1+r_2}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -4 & 9 \\ & -11 & 22 & -44 \end{array}\right)$$

$$\overset{(-1/11)r_2}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -4 & 9 \\ & 1 & -2 & 4 \end{array}\right)$$

Now backward substitution. Note that x_3 is free. Let $x_3 = t$.

$$\begin{aligned} x_2 &= 4 + 2t, \\ x_1 &= 9 - 2x_2 + 4x_3 = 1 \\ x &= \begin{pmatrix} 1 \\ 4+2t \\ t \end{pmatrix} \\ &= t\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} \\ &= x_H + x_P \end{aligned}$$

(2) Reduced row-echelon form:

$$\begin{pmatrix} 1 & 2 & -4 & | & 9 \\ 5 & -1 & 2 & | & 1 \end{pmatrix} \overset{(-5)r_1+r_2}{\longrightarrow} \begin{pmatrix} 1 & 2 & -4 & | & 9 \\ & -11 & 22 & | & -44 \end{pmatrix}$$
$$\overset{(-1/11)r_2}{\longrightarrow} \begin{pmatrix} 1 & 2 & -4 & | & 9 \\ & 1 & -2 & | & 4 \end{pmatrix}$$
$$\overset{(-2)r_2+r1}{\longrightarrow} \begin{pmatrix} 1 & & & | & 1 \\ & 1 & -2 & | & 4 \end{pmatrix}$$

Note that x_3 is free. Let $x_3 = t$.

$$\begin{aligned} x_2 &= 4 + 2t, \\ x_1 &= 1 \\ x &= \begin{pmatrix} 1 \\ 4+2t \\ t \end{pmatrix} \\ &= t\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} \\ &= x_H + x_P \end{aligned}$$

5. (1) Determine if $(A|b)$ is in row-echelon form, reduce it to a row-echelon form if not, and use its row-echelon form to solve $Ax = b$.
(2) Determine if $(A|b)$ is in reduced row-echelon form, reduce it to a THE reduced row-echelon form if not, and use THE row-echelon form to solve $Ax = b$.

$$(a) \begin{pmatrix} 1 & 0 & 0 & -1 & | & 1 \\ 0 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$
$$(b) \begin{pmatrix} 1 & 0 & 2 & -1 & | & 1 \\ 0 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$
$$(c) \begin{pmatrix} 0 & 0 & 1 & -1 & | & 1 \\ 1 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$
$$(d) \begin{pmatrix} 0 & 0 & 1 & -1 & | & 1 \\ 0 & 0 & 1 & 0 & | & 2 \\ 0 & 1 & 0 & -1 & | & 0 \end{pmatrix}$$

- **ans:**

(a) (1) Yes, in row-echelon form, rank is 2, two free variables, x_2 and x_4:

$$x = c_1\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

(2) Yes, in reduced row-echelon form, rank is 2, two free variables, x_2 and x_4:

$$x = c_1\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

(b) (1) Yes, in row-echelon form, rank is 2, two free variables, x_2 and x_4:

$$x = c_1\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

(2) No, not in reduced row-echelon form.
$-2r_2 + r_1$:

$$\begin{pmatrix} 1 & 0 & 0 & -1 & | & -3 \\ 0 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

rank is 2, two free variables, x_2 and x_4:

$$x = c_1\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

(c) (1) No, not in row-echelon form.
$r_1 \leftrightarrow r_2$:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & -1 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

rank is 2, two free variables, x_2 and x_4:

$$x = c_1\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

(2) No, not in reduced row-echelon form.
Continue above row operation, $-r_2 + r_1$:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & | & 1 \\ 0 & 0 & 1 & -1 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

rank is 2, two free variables, x_2 and x_4:

$$x = c_1\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

(d) (1) No, not in row-echelon form.

$r_1 \leftrightarrow r_3$:

$$\begin{pmatrix} 0 & 1 & 0 & -1 & | & 0 \\ 0 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & -1 & | & 1 \end{pmatrix}$$

$-r_2 + r_3$:

$$\begin{pmatrix} 0 & 1 & 0 & -1 & | & 0 \\ 0 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 0 & -1 & | & -1 \end{pmatrix}$$

$-r_3$:

$$\begin{pmatrix} 0 & 1 & 0 & -1 & | & 0 \\ 0 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 0 & 1 & | & 1 \end{pmatrix}$$

rank is 3, one free variable, x_1:

$$x = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

(2) No, not in reduced row-echelon form.
Continue above row operation, $r_3 + r_1$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & | & 1 \\ 0 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 0 & 1 & | & 1 \end{pmatrix}$$

rank is 3, one free variable, x_1:

$$x = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

6. Find the linearly dependence by rank and the number of free variables for the homogeneous system $Ax = 0$.

$$(1)\mathbf{u}_i = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix},$$

$$(2)\mathbf{u}_i = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$$

$$(3)\mathbf{u}_i = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

$$(4)\mathbf{u}_i = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$$

- **ans:** (1)

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 2 & 4 & 4 \end{pmatrix}$$
$$\rightarrow \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ & & 0 \end{pmatrix}$$

rank 2, one free variable x_2 in $Ax = 0$. Non zero solution, so they are linearly dependent.

(2)

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 2 & 4 \end{pmatrix}$$
$$\rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ & 0 \end{pmatrix}$$

rank 1, one free variable x_2 in $Ax = 0$. Non zero solution, so they are linearly dependent.

(3) They must be linearly dependent as more columns than the maximal possible rank 3.

$$A = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 2 \\ 2 & 4 & 4 & 4 \end{pmatrix}$$
$$\rightarrow \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 2 \\ & & 0 & 2 \end{pmatrix}$$

rank 3, one free variable (x_2) in $Ax = 0$. Non zero solution, so they are linearly dependent.

(4)

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 4 & 4 \end{pmatrix}$$
$$\rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ & & 4 \end{pmatrix}$$

rank 3, no free variable (3 columns) in $Ax = 0$. No non-zero solution, so they are linearly independent.

7. Find the determinant
(1) by cofactor expansions (no row/column operations)
(2) by row operations to an upper triangular matrix.
(3) by smart combinations of row/column op and expansion.

$$A = \begin{pmatrix} 2 & 0 & 4 & 4 \\ 5 & -10 & 5 & 5 \\ 0 & 0 & 4 & 4 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

- **ans:**

(1) Expansion by row 3

$$|A| = 4 \begin{vmatrix} 2 & 0 & 4 \\ 5 & -10 & 5 \\ 1 & 2 & 2 \end{vmatrix} - 4 \begin{vmatrix} 2 & 0 & 4 \\ 5 & -10 & 5 \\ 1 & 2 & 0 \end{vmatrix}$$

Then by row 1, for both:

$$|A| = 4(2\begin{vmatrix}-10 & 5\\ 2 & 2\end{vmatrix} + 4\begin{vmatrix}5 & -10\\ 1 & 2\end{vmatrix})$$
$$- 4(2\begin{vmatrix}-10 & 5\\ 2 & 0\end{vmatrix} + 4\begin{vmatrix}5 & -10\\ 1 & 2\end{vmatrix})$$
$$= 4(2(-30) + 4(20)) - 4(2(-10) + 4(20))$$
$$= -160$$

(2) row operations to an upper triangular matrix.

$$|A| \overset{\frac{1}{2}r_1}{=} 2\begin{vmatrix}1 & 0 & 2 & 2\\ 5 & -10 & 5 & 5\\ 0 & 0 & 4 & 4\\ 1 & 2 & 0 & 2\end{vmatrix}$$
$$\overset{\frac{1}{5}r_2}{=} 10\begin{vmatrix}1 & 0 & 2 & 2\\ 1 & -2 & 1 & 1\\ 0 & 0 & 4 & 4\\ 1 & 2 & 0 & 2\end{vmatrix}$$
$$\overset{\frac{1}{4}r_3}{=} 40\begin{vmatrix}1 & 0 & 2 & 2\\ 1 & -2 & 1 & 1\\ 0 & 0 & 1 & 1\\ 1 & 2 & 0 & 2\end{vmatrix}$$

$$|A| \overset{-r_1+r_2}{=} 40\begin{vmatrix}1 & 0 & 2 & 2\\ 0 & -2 & -1 & -1\\ 0 & 0 & 1 & 1\\ 1 & 2 & 0 & 2\end{vmatrix}$$
$$\overset{-r_1+r_4}{=} 40\begin{vmatrix}1 & 0 & 2 & 2\\ 0 & -2 & -1 & -1\\ 0 & 0 & 1 & 1\\ 0 & 2 & -2 & 0\end{vmatrix}$$
$$\overset{r_2\leftrightarrow r_4}{=} -40\begin{vmatrix}1 & 0 & 2 & 2\\ 0 & 2 & -2 & 0\\ 0 & 0 & 1 & 1\\ 0 & -2 & -1 & -1\end{vmatrix}$$
$$\overset{r_2+r_4}{=} -40\begin{vmatrix}1 & 0 & 2 & 2\\ 0 & 2 & -2 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & -3 & -1\end{vmatrix}$$

$$|A| \overset{3r_3+r_4}{=} -40\begin{vmatrix}1 & 0 & 2 & 2\\ 0 & 2 & -2 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 2\end{vmatrix}$$
$$= -40(1)(2)(1)(2) = -160$$

(3) Smart combination:

$$\begin{vmatrix}2 & 0 & 4 & 4\\ 5 & -10 & 5 & 5\\ 0 & 0 & 4 & 4\\ 1 & 2 & 0 & 2\end{vmatrix} \overset{-c_3+c_4}{=} \begin{vmatrix}2 & 0 & 4 & 0\\ 5 & -10 & 5 & 0\\ 0 & 0 & 4 & 0\\ 1 & 2 & 0 & 2\end{vmatrix}$$
$$\overset{\text{col 4 expan}}{=} 2\begin{vmatrix}2 & 0 & 4\\ 5 & -10 & 5\\ 0 & 0 & 4\end{vmatrix}$$
$$\overset{\text{row 3 expan}}{=} 2(4)\begin{vmatrix}2 & 0\\ 5 & -10\end{vmatrix}$$
$$= 2(4)(2)(-10) = -160$$

8. $|A_{4\times 4}| = -2$, $|B_{4\times 4}| = -1$, find
$$|2A^{-1}|,\ |A^TB|,\ |-2B^{-1}|,\ |B^2|$$

- **ans:** Factor 2 out of each row:
$$|2A^{-1}| = 2^4|A|^{-1} = -8$$
$$|A^TB| = |A^T||B| = |A||B| = 2$$
$$|-2B^{-1}| = (-2)^4|B^{-1}| = (-2)^4|B|^{-1} = -16$$
$$|B^2| = |B||B| = 1$$

9. Find A^{-1} by (1) cofactors, (2) row operations. And use A^{-1} to solve $Ax = b$.
$$(a)\ A = \begin{pmatrix}2 & 5\\ 1 & 3\end{pmatrix}, \qquad b = \begin{pmatrix}5\\ 3\end{pmatrix}$$
$$(b)\ A = \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix} \qquad b = \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$$

- **ans:** (a) (1) Cofactor method:

$$A^{-1} = \frac{1}{|A|}C^T$$
$$= \frac{1}{|A|}\begin{pmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}^T$$
$$= \frac{1}{\begin{vmatrix}2 & 5\\ 1 & 3\end{vmatrix}}\begin{pmatrix}3 & -(1)\\ -(5) & 2\end{pmatrix}^T$$
$$= \frac{1}{1}\begin{pmatrix}3 & -1\\ -5 & 2\end{pmatrix}^T = \begin{pmatrix}3 & -5\\ -1 & 2\end{pmatrix}$$

(a) (2) Row operation method:
$$(A\ I) = \begin{pmatrix}2 & 5\\ 1 & 3\end{pmatrix}, \begin{pmatrix}1 & \\ 0 & 1\end{pmatrix}$$
$$\overset{(-1/2)r_1+r_2}{\longrightarrow} \begin{pmatrix}2 & 5\\ & 0.5\end{pmatrix}, \begin{pmatrix}1 & \\ -0.5 & 1\end{pmatrix}$$
$$\overset{2r_2;(1/2)r_1}{\longrightarrow} \begin{pmatrix}1 & 2.5\\ & 1\end{pmatrix}, \begin{pmatrix}0.5 & \\ -1 & 2\end{pmatrix}$$
$$\overset{(-2.5)r_2+r_1}{\longrightarrow} \begin{pmatrix}1 & \\ & 1\end{pmatrix}, \begin{pmatrix}3 & -5\\ -1 & 2\end{pmatrix}$$
$$= (I\ A^{-1})$$

$$A^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

(a) Solution:

$$x = A^{-1}b = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(b) (1) Cofactor method:

$$\begin{aligned}
A^{-1} &= \frac{1}{|A|} C^T \\
&= \frac{1}{|A|} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}^T \\
&= \frac{1}{\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}} \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \end{pmatrix}^T \\
&= \frac{1}{1} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

(b)(2) Row operation method:

$$\begin{aligned}
(A\ I) &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ 0 & 1 & \\ & & 1 \end{pmatrix} \\
&\overset{-r2+r_1}{\rightarrow} \begin{pmatrix} 1 & & \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & \\ 0 & 1 & \\ & & 1 \end{pmatrix} \\
&\overset{-r_3+r_2}{\rightarrow} \begin{pmatrix} 1 & & \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & \\ 0 & 1 & -1 \\ & & 1 \end{pmatrix} \\
&= (I\ A^{-1})
\end{aligned}$$

$$A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution:

$$x = A^{-1}b = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

10. Solve $Ax = b$ by Cramer's rule

(1) $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(2) $A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix}$

- **ans:**

$$x_k = \frac{|B_k|}{\det A}$$

where B_k is obtained by replacing k-th column of A.

(1)

$$\begin{aligned}
x &= \frac{1}{\det A} \begin{pmatrix} \det B_1 \\ \det B_2 \end{pmatrix} \\
&= \frac{1}{\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}} \begin{pmatrix} \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}
\end{aligned}$$

(2)

$$\begin{aligned}
x &= \frac{1}{\det A} \begin{pmatrix} \det B_1 \\ \det B_2 \\ \det B_3 \end{pmatrix} \\
&= \frac{1}{\begin{vmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{vmatrix}} \begin{pmatrix} \begin{vmatrix} -2 & 0 & 4 \\ 0 & 1 & 2 \\ -2 & 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 2 & -2 & 4 \\ 0 & 0 & 2 \\ 1 & -2 & 3 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{vmatrix} \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}
\end{aligned}$$

11. Find eigenvalues and eigenvectors.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{pmatrix}$$

- **ans:**

$$\det(A - \lambda I) = 0, \ \lambda = 1, 4, -2$$

$\lambda = 1$

$$A - \lambda I = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 5 & -2 \end{pmatrix}, \ x = C \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = 4$

$$A - \lambda I = \begin{pmatrix} -3 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \end{pmatrix}, \ x = C \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = -2$

$$A - \lambda I = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 5 & 1 \end{pmatrix}, \ x = C \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix}$$

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a=[1 2 1;0 3 1;0 5 -1];
[p e]=eig(a)
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12. Find eigenvalues and eigenvectors.

$$\begin{pmatrix} -1 & 2 \\ -5 & 1 \end{pmatrix}$$

- **ans:**

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 2 \\ -5 & 1-\lambda \end{vmatrix},$$
$$\lambda = \pm 3i$$

$\lambda = 3i$

$$A - \lambda I = \begin{pmatrix} -1-3i & 2 \\ -5 & 1-3i \end{pmatrix}$$
$$\to \begin{pmatrix} -1-3i & 2 \\ & 0 \end{pmatrix}$$

Choose the opposite coeffcients with one negative sign:

$$x = C\begin{pmatrix} 2 \\ 1+3i \end{pmatrix}$$

$\lambda = -3i$

$$A - \lambda I = \begin{pmatrix} -1+3i & 2 \\ -5 & 1+3i \end{pmatrix}$$
$$\to \begin{pmatrix} -1+3i & 2 \\ & 0 \end{pmatrix}$$

Choose the opposite coeffcients with one negative sign:

$$x = C\begin{pmatrix} 2 \\ 1-3i \end{pmatrix}$$

Note that the two roots are conjugate, and the two eigenvectors are conjugate.

13. Find Orthogonal eig-matrices:

$$(1)\ A = \begin{pmatrix} 1 & 9 \\ 9 & 1 \end{pmatrix}$$
$$(2)\ A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- **ans:** (1) Find eigenvalues and eigenvectors.

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 9 \\ 9 & 1-\lambda \end{vmatrix}$$
$$= (1-\lambda)^2 - 9^2 = 0$$

$$\lambda = -8, 10$$

For $\lambda = -8$,

$$A - \lambda I = \begin{pmatrix} 9 & 9 \\ 9 & 9 \end{pmatrix},\ K_1 = c\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda = 10$,

$$A - \lambda I = \begin{pmatrix} -9 & 9 \\ 9 & -9 \end{pmatrix},\ K_2 = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The second vector is orthogonal to the first vector:

$$K_2' K_1 = 0$$

We only need to normalize them:

$$\mathbf{w}_1 = K_1/\|K_1\| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\mathbf{w}_2 = K_2/\|K_2\| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Checking: $P^T P = I$.

(2) Find eigenvalues and eigenvectors.

$$|A-\lambda I| = \begin{vmatrix} -\lambda & -1 & 0 \\ -1 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix}$$
$$= -\lambda\begin{vmatrix} -1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 0 & -\lambda \end{vmatrix}$$
$$= -\lambda(\lambda + \lambda^2 - 1) + \lambda$$
$$= -\lambda(\lambda + \lambda^2 - 2)$$
$$= -\lambda(\lambda - 1)(\lambda + 2) = 0$$

$$\lambda = 0, 1, -2$$

For $\lambda = 0$, solve $(A - \lambda I)x = 0$

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ & & 0 \end{pmatrix},$$
$$K_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda = 1$, solve $(A - \lambda I)x = 0$

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & -1 & 0 \\ & -1 & 1 \\ & & 0 \end{pmatrix},$$
$$K_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda = -2$, solve $(A - \lambda I)x = 0$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 0 \\ & 1/2 & 1 \\ & & 0 \end{pmatrix},$$
$$K_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$v_1 = K_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$v_2 = K_2 - \text{Proj}_{v_1} K_2 = K_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$v_3 = K_3 - \text{Proj}_{v_1} K_3 - \text{Proj}_{v_2} K_3$$

$$= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\mathbf{w}_1 = K_1 / \|K_1\| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{w}_2 = K_2 / \|K_2\| = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{w}_3 = K_3 / \|K_3\| = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{pmatrix}$$

Checking: $P^T P = I$.

14. Factor the matrix A into a product XDX^{-1}, where D is diagonal. Then find A^{132} by the factorization.

$$(a)\ A = \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix};$$

$$(b)\ A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

- **ans:** (a) Find eigenvalues:

$$\det(A - \lambda I) = 0,\ \lambda = -1, 0$$

For $\lambda = -1$, solve $(A - \lambda I)x = 0$:

$$\begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix},\ x = c \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

For $\lambda = 0$, solve $(A - \lambda I)x = 0$:

$$\begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix},\ x = c \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$X = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$X^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$$

$$A = XDX^{-1}$$

$$= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$$

$$A^{132} = XD^{132}X^{-1}$$

$$= \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} (-1)^{132} & 0 \\ 0 & 0^{132} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}$$

(b) Find eigenvalues:

$$\det(A - \lambda I) = 0,\ \lambda = 1, 1, -1$$

$\lambda = 1$

$$A - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ x = C \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ C \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = -1$

$$A - \lambda I = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix},\ x = C \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix},$$

$$X^{-1} = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 1/2 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

$$A = XDX^{-1}$$

$$A^{132} = XD^{132}X^{-1} = XIX^{-1} = I = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

15. Find possible values for a that make the matrix defective, i.e., A is not diagonalizible, or show that no such values exist.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$$

- **ans:** Find eigenvalues:

$$\det(A - \lambda I) = 0,\ \lambda = 0, 2, a$$

Three cases:

(1) If $a \neq 0, 2$, then we have 3 different eigenvalues, and the matrix would not be defective. So A is diagonalizible.

(2) Let us check $a = 0$. We have repeated eigenvalue 0. $\lambda = 0, 0$

$$A - \lambda I = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; x = C \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; C \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

We do have two linearly independent eigenvectors. So A is not defective. So A is diagonalizible.

(3) Let us check $a = 2$. We have repeated eigenvalue 2. $\lambda = 2, 2$

$$A - \lambda I = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \; x = C \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We have only one linearly independent eigenvector. So A is defective. So A is not diagonalizible.

Therefore, $a = 2$ is the only value for which A is not diagonalizible.

```
a=2
A=[1 1 0; 1 1 0; 0 0 a];
[x e]=eig(A)
A*x, x*e
```