M351 Study Guide 2 (S. Zhang) .

1. Solve
$$y'' - 16y = 2e^{4x}$$
   - **ans:**

   For y_H, char equation
$$r^-16 = 0$$
   Roots are
$$r = \pm 4$$
$$y_H = c_1 e^{4x} + c_2 e^{-4x}$$
   Because 4 is a root, we have <u>an extra</u> x in y_p:
$$y_p = x(Ae^{4x})$$
   Plug it into equation, we fond $A = 1/4$.
$$y = y_H + y_p = c_1 e^{4x} + c_2 e^{-4x} + \frac{1}{4} x e^{4x}$$

2. Solve
$$y'' - 2y' - 3y = 3e^{2t}.$$
   - **ans:** Characteristic equation is
$$r^2 - 2r - 3 = 0, \; r = -1, 3$$
$$y_H = c_1 e^{-t} + c_2 e^{3t}$$
   $r \neq 2$,
$$y_P = Ae^{2t}$$
   Plug y_P into DE:
$$A = -1$$
$$y = y_H + y_P = c_1 e^{-t} + c_2 e^{3t} - e^{2t}$$

3. Solve IVP
$$y'' + y = 2 \sin t, \quad y(0) = 0, \; y'(0) = -1.$$
   - **ans:** Characteristic equation is
$$r^2 + 1 = 0, \; r = \pm i$$
$$y_H = c_1 \cos t + c_2 \sin t$$
$$y_P = (A \cos t + B \sin t)t$$
   Plug y_P into DE:
$$A = -1, \; B = 0$$
$$y = y_H + y_P = c_1 \cos t + c_2 \sin t - t \cos t$$
   By IVP
$$y = -t \cos t$$

4. Find the general solution of the <u>homogeneous</u> equation y_c and <u>write down</u> the form for undetermined coefficients for y_p. (Do not solve for y_p.)
$$(D^2 - 9)(D + 3)(D^2 + 4)y = e^x \sin x - e^{-3x} + \cos 2x.$$
   - **ans:**
$$r = 3, -3, -3, \pm 2i$$
$$y_c = y_H = e^{-3x}(A + Bx) + e^{3x}(E) + ((G) \cos 2x + (I) \sin 2x.$$
$$y_p = e^x (A \cos x + B \sin x) + (C) x^2 e^{-3x} + x(E \cos 2x + F \sin 2x)$$
   Note the extra x and x^2 above.

5. Find the general solution by both the undetermined coefficients method and the variation of parameters method.
$$y'' - y = \cos^2 x.$$
   - **ans:** For y_c, $r^2 - 1 = 0$,
$$y_c = C_1 e^x + C_2 e^{-x}.$$
   For y_p by UC, we need to rewrite $f = \cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2}$
$$y_p = A + B \cos 2x + C \sin 2x$$
$$\Rightarrow$$
$$-4B \cos 2x - 4C \sin 2x - A - B \cos 2x + C \sin 2x$$
$$= \frac{1}{2} + \frac{\cos 2x}{2}$$
$$\Rightarrow \; y_p = -\frac{1}{2} - \frac{\cos 2x}{10}$$
   The solution is
$$y = y_c + y_p = C_1 e^x + C_2 e^{-x} - \frac{1}{2} - \frac{\cos 2x}{10}$$
   For y_p by VP,
$$y_p = u_1 e^x + u_2 e^{-x}$$
$$u_1' e^x + u_2' e^{-x} = 0$$
$$u_1' e^x - u_2' e^{-x} = \cos^2 x$$
   Add the two equations,
$$u_1 = \int \frac{1}{2} e^{-x} \cos^2 x$$
   To find the integral, we still need to rewrite $\cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2}$. The first integral is $\int \frac{1}{4} e^{-x} = -\frac{1}{4} e^{-x}$. The second integral needs integration by parts twice. Let us see.
$$u = \cos 2x, dv = e^{-x} dx$$
$$du = -2 \sin 2x dx, v = -e^{-x}$$

$$\int e^{-x}\cos 2x = -e^{-x}\cos 2x - \int 2e^{-x}\sin 2x dx$$

Once again

$$u = \sin 2x, dv = e^{-x}dx$$
$$du = 2\cos 2x dx, v = -e^{-x}$$

$$\int e^{-x}\cos 2x = -e^{-x}\cos 2x$$
$$+2e^{-x}\sin 2x - 4\int e^{-x}\cos 2x$$

Move the integral term to left,

$$\int e^{-x}\cos 2x = -\frac{1}{5}e^{-x}\cos 2x + \frac{2}{5}e^{-x}\sin 2x$$

Therefore,

$$u_1 = -\frac{1}{4}e^{-x} - \frac{1}{20}e^{-x}\cos 2x + \frac{1}{10}e^{-x}\sin 2x$$

By the first equation of the two equations for $u_1'\&u_2'$,

$$\begin{aligned} u_2' &= -e^{2x}u_1' \\ &= -e^{2x}\left(\frac{1}{4}e^{-x} + \frac{1}{4}e^{-x}\cos 2x\right) \\ &= -\frac{1}{4}e^{x} - \frac{1}{4}e^{x}\cos 2x \\ u_2 &= -\frac{1}{4}e^{x} - \frac{1}{4}\int e^{x}\cos 2x \end{aligned}$$

Well, we need integration by parts twice again.

$$\begin{aligned} &\int e^{x}\cos 2x dx \\ =&e^{x}\cos 2x + 2\int e^{x}\sin 2x dx \\ =&e^{x}\cos 2x + 2e^{x}\sin 2x - 4\int e^{x}\cos 2x dx \end{aligned}$$

$$\int e^{x}\cos 2x dx = \frac{1}{5}e^{x}\cos 2x + \frac{2}{5}e^{x}\sin 2x$$

Therefore,

$$u_2 = -\frac{1}{4}e^{x} - \frac{1}{20}e^{x}\cos 2x - \frac{1}{10}e^{x}\sin 2x$$

Finally $y = y_c + y_p$ and (the same as the one by UC)

$$\begin{aligned} y_p =& -\frac{1}{4} - \frac{1}{20}\cos 2x + \frac{1}{10}\sin 2x \\ &-\frac{1}{4} - \frac{1}{20}\cos 2x - \frac{1}{10}\sin 2x \\ =& -\frac{1}{2} - \frac{1}{10}\cos 2x \end{aligned}$$

6. Solve

$$y'' + y = \tan x$$

- **ans:** For $y_c = y_H$,

$$r^2 + 1 = 0, \; r = \pm i$$

$$y_H = c_1\cos x + c_2\sin x$$

For y_p by VP, (note that the method of undetermined coefficients won't work here as $\tan x$ is not one of those special functions),

$$y_p = u_1y_1 + u_2y_2$$

satisfies eqautions:

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= f \end{aligned}$$

Here we have

$$\begin{aligned} u_1'\cos x + u_2'\sin x &= 0 \\ -u_1'\sin x + u_2'\cos x &= \tan x \end{aligned}$$

$(equation1) * \cos x - (equation2) * \sin x$:

$$\begin{aligned} u_1' &= -\tan x\sin x \\ u_1 &= \int -\tan x\sin x = \int -\frac{\sin^2 x}{\cos x} \\ &= \int \frac{\cos^2 x - 1}{\cos x} = \int(\cos x - \sec x) \\ &= \int \cos x dx - \int \frac{\sec x(\sec x + \tan x)}{\sec x(\sec x + \tan x)}dx \\ &= \sin x - \ln(\sec x + \tan x) \end{aligned}$$

$(equation1) * \sin x - (equation2) * \cos x$:

$$\begin{aligned} u_2' &= \tan x\cos x = \sin x \\ u_2 &= \int \sin x dx = -\cos x \end{aligned}$$

The general solution:

$$\begin{aligned} y = y_H + y_P =& c_1\cos x + c_2\sin x \\ &+ (\sin x + \ln(\sec x + \tan x))\cos x \\ &+ (-\cos x)\sin x \\ =& c_1\cos x + c_2\sin x - \cos x\ln(\sec x + \tan x) \end{aligned}$$

Note that we can use the following formula to solve the two linear equations above:

$$\begin{aligned} u_1' &= \frac{-y_2 f}{W(y_1, y_2)} \\ u_2 &= \frac{y_1 f}{W(y_1, y_2)} \end{aligned}$$

Here

$$W(\cos x, \sin x) = 1$$

$$u_1 = \int \frac{-\sin x \tan x}{1} dx$$
$$u_2 = \int \frac{\cos x \tan x}{1} dx.$$

Then repeating above work.

7. Solve

$$y''' + y' = \tan x$$

- **ans:** For $y_c = y_H$,

$$r^3 + r = 0, \ r = 0, \pm i$$

$$y_H = c_1 + c_2 \cos x + c_3 \sin x$$

For y_p by VP, (note that the method of undetermined coefficients won't work here as $\tan x$ is not one of those special functions),

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$$

satisfies eqautions:

$$\begin{aligned} u_1' y_1 + u_2' y_2 + u_3' y_3 &= 0 \\ u_1' y_1' + u_2' y_2' + u_3' y_3' &= 0 \\ u_1' y_1'' + u_2' y_2'' + u_3' y_3'' &= q \end{aligned}$$

Here we have

$$\begin{aligned} u_1' + u_2' \cos x + u_3' \sin x &= 0 \\ -u_2' \sin x + u_3' \cos x &= 0 \\ -u_2' \cos x - u_3' \sin x &= \tan x \end{aligned}$$

$(equation2) * \sin x + (equation3) * \cos x$:

$$\begin{aligned} -u_2' &= \tan x \cos x \\ u_2 &= \int -\tan x \cos x = -\int \sin x dx = \cos x \end{aligned}$$

$(equation2) * \cos x - (equation2) * \sin x$:

$$\begin{aligned} u_3' &= -\tan x \sin x \\ u_3 &= \int -\frac{\sin^2 x}{\cos x} \\ &= \int \frac{\cos^2 x - 1}{\cos x} = \int (\cos x - \sec x) \\ &= \int \cos x dx - \int \frac{\sec x(\sec x + \tan x)}{\sec x(\sec x + \tan x)} dx \\ &= \sin x - \ln(\sec x + \tan x) \end{aligned}$$

By $(equation1)$:

$$\begin{aligned} u_1' &= -u_2' \cos x - u_3' \sin x \\ &= \tan x \cos x \cos x + \tan x \sin x \sin x \\ &= \tan x \\ u_1 &= \int \tan x = -\ln \cos x \end{aligned}$$

The general solution:

$$\begin{aligned} y = y_H + y_P =& c_1 + c_2 \cos x + c_3 \sin x \\ & - \ln \cos x \\ & + (\cos x) \cos x \\ & + (\sin x + \ln(\sec x + \tan x)) \sin x \\ =& c_1' + c_2 \cos x + c_3 \sin x \\ & - \ln \cos x - \sin x \ln(\sec x + \tan x)) \end{aligned}$$

Note that we can use the formula in the book for solving the linear system of 3 equations above. But it takes more time to compute 4 determinants that way.

8. Solve an Euler equation:

$$(x-1)^2 y'' + 8(x-1)y' + 12y = 0$$

- **ans:**

$$r(r-1) + 8r + 12 = 0$$

$$r = -3, -4$$

$$y = c_1 (x-1)^{-3} + c_2 (x-1)^{-4}$$

9. Solve an Euler equation:

$$x^2 y'' - 5xy' + 9y = 0$$

- **ans:**

$$r(r-1) - 5r + 9 = 0$$

$$r = 3, 3$$

$$y = c_1 x^3) + c_2 x^3 \ln x$$

10. Solve an Euler equation:

$$x^2 y'' + 3xy' - 4y = 0$$

- **ans:**

$$r(r-1) + 3r - 4 = 0$$

$$r = -1 \pm \sqrt{5}$$

$$y = c_1 x^{-1+\sqrt{5}}) + c_2 x^{-1-\sqrt{5}}$$

11. Solve an Euler equation:

$$25x^2y'' + 25xy' + y = 0$$

- **ans:**

$$25r(r-1) + 25r + 1 = 0$$

$$r = \pm\frac{1}{5}i$$

$$y = c_1\sin(\frac{1}{5}\ln x)) + c_2\sin(\frac{1}{5}\ln x)$$

12. Solve a non homogeneous Euler equation:

$$2x^2y'' + 5xy' + y = x^2 - x$$

- **ans:** For $y_H = y_c$:

$$2r(r-1) + 5r + 1 = 0$$

$$r = -1, -\frac{1}{2}$$

$$y = c_1\frac{1}{x} + c_2\frac{1}{\sqrt{x}}$$

For y_p, we have to use the method of variation of parameter as we have a non-constant coefficient equation, the method of undeteminend coefficients does not work.

However, we have to normalize the equation first!

$$y'' + \frac{5}{2x}y' + \frac{1}{2x^2}y = \frac{1}{2} - \frac{1}{2x}$$

$$y_p = u_1y_1 + u_2y_2$$

$$u_1'x^{-1} + u_2'x^{-1/2} = 0$$
$$-u_1'x^{-2} - \frac{1}{2}u_2'x^{-3/2} = \frac{1}{2} - \frac{1}{2x}$$

$(1/2)eq1/x + eq2$:

$$u_1'(\frac{1}{2} - 1)x^{-2} = \frac{1}{2} - \frac{1}{2x}$$
$$u_1' = (-1 + \frac{1}{x})x^2$$
$$u_1 = -\frac{1}{3}x^3 + \frac{1}{2}x^2$$

$eq1/x + eq2$:

$$u_2'(1 - \frac{1}{2})x^{-3/2} = \frac{1}{2} - \frac{1}{2x}$$
$$u_2' = (1 - \frac{1}{x})x^{3/2}$$
$$u_2 = \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}$$

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 \\ &= (-\frac{1}{3}x^2 + \frac{1}{2}x) + (\frac{2}{5}x^2 - \frac{2}{3}x) \\ &= \frac{1}{15}x^2 - \frac{1}{6}x \end{aligned}$$

The general solution is

$$y = y_H + y_p = c_1\frac{1}{x} + c_2\frac{1}{\sqrt{x}} + \frac{1}{15}x^2 - \frac{1}{6}x$$

13. Taylor series solution of IVP:

$$y'' + y^2 = 1$$
$$y(0) = 2, \quad y'(0) = 3$$

- **ans:**

$$y'' = -y^2 + 1 \qquad y''(0) = -2^2 + 1 = -3$$

$$y''' = -2yy' \qquad y'''(0) = -12$$

$$y^{(4)} = -2yy'' - 2(y')^2 \qquad y^{(4)}(0) = -6$$

$$y^{(5)} = -2yy''' - 6(y')y'' \qquad y^{(4)}(0) = 102$$

Taylor formula:

$$y = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + ...$$

$$y = 2 + 3x - \frac{3}{2}x^2 - 2x^3 - \frac{1}{4}x^4 + \frac{17}{20}x^5$$

14. Solve the equation by 3 methods. (A) As a constant coefficient equation, (B) By reduction of order method, type no-y, (C) By reduction of order method, type no-x.

$$3y'' - 2y' = 0$$

- **ans:** (A)

$$3r^2 - 2r = 0, \; r = 0, 2/3$$

$$y = y_H = c + de^{2x/3}$$

(B) $u = y'$

$$3u' - 2u = 0$$

$$3\frac{du}{u} = 2dx$$

$$3\ln u = 2x + C$$

$$u = ce^{2x/3}$$

$$y = \int u = c\int e^{2x/3}$$
$$= c(\frac{3}{2}e^{2x/3} + d) = c_1 e^{2x/3} + c_2$$

(C) $u = y'$, but

$$y'' = u\frac{du}{dy}$$

$$3u\frac{du}{dy} - 2u = 0$$

Divided by u (may check additional solution $u = 0$ below.)

$$3du = 2dy$$

$$3u = 2y + c_1$$

$$3\frac{dy}{dx} = 2y + c_1$$

$$\int \frac{3dy}{2y + c_1} = \int dx$$

$$\frac{3}{2}\ln(2y + c_1) = x + c_2$$

$$2y + c_1 = e^{\frac{2x}{3} + \frac{2}{3}c_2}$$

$$y = -\frac{1}{2}c_1 + \frac{1}{2}e^{\frac{2}{3}c_2}e^{2x/3}$$

$$y = C_1 + C_2 e^{2x/3}$$

15. Solve the nonlinear equation by reduction of order:

$$yy'' = (y')^2$$

- **ans:** No x, $u = y'$, $du/dy = (dy'/dx)(dy/dx)$.

check $u = 0$.

$$\frac{du}{u} = \frac{dy}{y}$$

$$u = c_1 y$$

$$\frac{dy}{y} = c_1 y$$

$$y = c_2 e^{c_1 x}$$

16. A mass of 4 pounds is attached to a spring whose constant is 2 lb/ft. The medium offers a damping force that is numerically equal to the instantaneous velocity. The mass is initially released from a point 1 foot above the equilibrium position with a downward velocity of 8 ft/s.

Determine the time at which the mass passes through the equilibrium position.

Find the time at which the mass attains its extreme displacement from the equailibrium position. What is the position of mass at this instant..

- **ans:** Change the mass unit:

$$m = \frac{4}{32} = \frac{1}{8}\text{lb-sec}^2/\text{ft}$$

The gravity cancels a stretch of spring. We have only two forces, spring force and resistance force:

$$mx'' + \gamma x' + kx = 0$$
$$\frac{1}{8}x'' + x' + 2x = 0$$
$$\frac{1}{8}r^2 + r + 2 = 0$$
$$r = -4, -4$$

$$x(t) = e^{-4t}(A + Bt)$$
$$x(0) = -1, \ -1 = A$$
$$x'(0) = 8, \ 4 = B$$
$$x(t) = e^{-4t}(-1 + 4t)$$

First time passing equilibrium point, $x(t) = 0$

$$0 = -1 + 4, \quad t = 1/4$$

$$x'(t) = e^{-4t}(4 - 16t + 4)$$
$$x'(t) = 0,$$
$$0 = 8 - 16t, \quad t = 1/2$$
$$x(1/2) = e^{-4/2}(-1 + 4/2) = e^{-2} = 0.135$$

17. Find the charge on the capacitor in an LRC-series circuit when

$$L = \frac{1}{2}h, \ R = 10\Omega, \ C = 0.01f, \ E = 150V$$

with initial condition

$$q(0) = 1C, \ i(0) = 0A.$$

What is the charge on the capacitor after a long time?

- **ans:**

$$LQ'' + RQ' + C^{-1}Q = E$$
$$\frac{1}{2}Q'' + 10Q' + 100Q = 150$$

Find Q_H:

$$\frac{1}{2}r^2 + 10r + 100 = 0,\ r = -10 \pm 10i$$

$$Q_H = e^{-10t}(A\cos 10t + B\sin 10t)$$

Find Q_P:

$$Q_P = C\ \Rightarrow C = \frac{3}{2}$$

$$Q = Q_H + Q_P = e^{-10t}(A\cos 10t + B\sin 10t) + \frac{3}{2}$$

By

$$Q(0) = 1,\ Q'(0) = 0$$

$$Q(t) = -\frac{1}{2}e^{-10t}(\cos 10t + \sin 10t) + \frac{3}{2}$$

When $t \to \infty$, $e^{-10t} \to 0$

$$Q(t) \to \frac{3}{2}C$$

18. Find linearly dependence

(1) $\mathbf{u}_1 = \langle 0,1,1\rangle,\ \mathbf{u}_2 = \langle 1,2,0\rangle,\ \mathbf{u}_3 = \langle -1,0,2\rangle,$
(2) $\mathbf{u}_1 = x + x^2,\ \mathbf{u}_2 = 1 + 2x,\ \mathbf{u}_3 = -1 + 2x^2,$

- **ans:** For both problems, we look for nonzero solutions:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$$

We derive the same system of three equations for both problems:

$$\begin{aligned} c_2 \quad -c_3 &= 0 \\ c_1 + 2c_2 \quad &= 0 \\ c_1 \quad +2c_3 &= 0 \end{aligned}$$

Let $c_3 = -1$, we get a set of nonzero solutions:

$$2\mathbf{u}_1 - \mathbf{u}_2 - \mathbf{u}_3 = 0$$

So the answer is linearly dependent for both problems.

19. Given three vectors:

$$\mathbf{u}_1 = \langle 0,1,0\rangle,\ \mathbf{u}_2 = \langle 1,2,0\rangle,\ \mathbf{u}_3 = \langle 1,1,3\rangle,$$

(a) Show linearly independence
(b) Find a linear combination for $\mathbf{a} = \langle 0,1,0\rangle$, i.e, coordinates of $\mathbf{a}$, under the basis, $\mathbf{u}_1, \ldots.$
(c) Find the orthgonal bases by the Gram-Schmidt orthogonalization process, $\mathbf{v}_1$ ....
(d) Find the orthnormal bases by the Gram-Schmidt orthogonalization process, $\mathbf{w}_1$ ...
(e) Find a linear combination for $\mathbf{a} = \langle 0,1,0\rangle$, i.e, coordinates of $\mathbf{a}$, under the orthnormal basis above $\mathbf{w}_1$ ....

- **ans:**

(a) Only zero solutions:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$$

$$\begin{aligned} c_2 \quad +c_3 &= 0 \\ c_1 + 2c_2 + c_3 &= 0 \\ 3c_3 &= 0 \end{aligned}$$

$c_1 = c_2 = c_3 = 0.$

(b) Solve linear system:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{a}$$

$$\begin{aligned} c_2 \quad +c_3 &= 0 \\ c_1 + 2c_2 + c_3 &= 1 \\ 3c_3 &= 0 \end{aligned}$$

to get

$$c_1 = 1,\ c_2 = 0,\ c_3 = 0$$

$$\langle 0,1,0\rangle = (1)\mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3$$

Coordinates $(1,0,0)$.

(c) Orthogonal basis.

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1 = \langle 0,1,0\rangle \\ \mathbf{v}_2 &= \mathbf{u}_2 - proj_{\mathbf{v}_1}\mathbf{u}_1 = \langle 1,0,0\rangle \\ \mathbf{v}_3 &= \mathbf{u}_3 - proj_{\mathbf{v}_1}\mathbf{u}_3 - proj_{\mathbf{v}_2}\mathbf{u}_3 = \langle 0,0,3\rangle \end{aligned}$$

(d) Orthonormal basis.

$$\mathbf{w}_i = \mathbf{v}_i/\|\mathbf{v}_i\|$$

$$\begin{aligned} \mathbf{w}_1 &= \langle 0,1,0\rangle \\ \mathbf{w}_2 &= \langle 1,0,0\rangle \\ \mathbf{w}_3 &= \langle 0,0,1\rangle \end{aligned}$$

(e) The coordinates under $\mathbf{w}_i$.
2 method:
(1) solving lienar system as in (b):

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{a}$$

$$
\begin{aligned}
c_2 \qquad &= 0 \\
c_1 \qquad\qquad &= 1 \\
c_3 &= 0
\end{aligned}
$$

to get

$$c_1 = 1,\ c_2 = 0,\ c_3 = 0$$

$$\langle 0, 1, 0\rangle = (1)\mathbf{w}_1 + 0\mathbf{w}_2 + 0\mathbf{w}_3$$

Coordinates $(1, 0, 0)$.
(2) using inner products.

$$c_1 = \mathbf{a}\cdot\mathbf{w}_1 = 1$$

$$c_2 = \mathbf{a}\cdot\mathbf{w}_2 = 0$$

$$c_3 = \mathbf{a}\cdot\mathbf{w}_3 = 0$$

$$\mathbf{a} = 1\mathbf{w}_1 + 0\mathbf{w}_2 + 0\mathbf{w}_3$$