

M351 Study Guide 1 (S. Zhang) .

1. Verify that the piecewise defined function

$$y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

is a solution of the differential equation

$$xy' - 2y = 0 \quad \text{on } (-\infty, \infty).$$

• **ans:** We have to check the function in three parts of x region.

(1) $x < 0$,

$$\begin{aligned} y &= -x^2 \\ y' &= -2x \\ x(-2x) - 2(-x^2) &= 0 \end{aligned}$$

(2) $x = 0$,

$$\begin{aligned} y(0) &= (x^2)_{x=0} = 0 \\ y'_+(0) &= \lim_{x \rightarrow 0^+} y'(x) = \lim_{x \rightarrow 0^+} 2x = 0 \\ y'_-(0) &= \lim_{x \rightarrow 0^-} y'(x) = \lim_{x \rightarrow 0^-} -2x = 0 \\ y'(0) &= y'_+(0) = y'_-(0) = 0 \end{aligned}$$

differential equation becomes to

$$\begin{aligned} xy' - 2y &= 0 \\ 0(0) - 2(0) &= 0 \end{aligned}$$

(3) $x > 0$,

$$\begin{aligned} y &= x^2 \\ y' &= 2x \\ x(2x) - 2(x^2) &= 0 \end{aligned}$$

2. Solve IVP with the general solution $y = C_1 e^x + C_2 e^{-x}$:

$$y'' - y = 0, y(0) = 1, y'(0) = 2$$

• **ans:**

$$\begin{aligned} y(0) = 1 &\Rightarrow 1 = C_1 + C_2 \\ y' = C_1 e^x - C_2 e^{-x} \\ y'(0) = 2 &\Rightarrow 2 = C_1 - C_2 \\ \Rightarrow C_1 &= \frac{3}{2}, C_2 = -\frac{1}{2} \\ y &= \frac{3}{2}e^x - \frac{1}{2}e^{-x} \end{aligned}$$

3. Without solving the equation, find and plot the region in xy plane for the following equation such that there is a unique solution through each given initial point in this region

$$y' = \sqrt{xy}$$

• **ans:**

$$\begin{aligned} f &= \sqrt{xy} \Rightarrow xy \geq 0 \\ f_y &= \frac{1}{2} \sqrt{\frac{x}{y}} \Rightarrow y \neq 0, xy > 0 \end{aligned}$$

For any point (x_0, y_0) in the first and the third quadrant, excluding $y = 0$ line, we have locally a unique solution for IV $y(x_0) = y_0$.

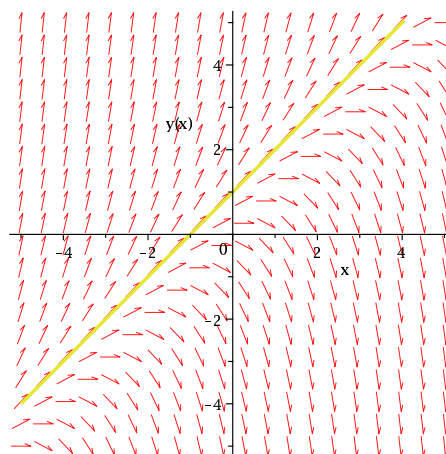
4. Make a table for y' and plot the direction field, including at least points $(0, 1)$, $(1, 1)$ and $(1, 0)$. Then sketch the solution curve corresponding to the given initial condition. Finally, use this solution curve to estimate the desired value of the solution:

$$\frac{dy}{dx} = y - x; \quad y(0) = 1, \quad y(3) = ?$$

• **ans:**

(x, y)	$(0, 1)$	$(1, 1)$	$(1, 0)$
$y' = y - x$	1	0	-1

We draw a little arrow at point $(0, 1)$ with a slope $y' = 1$.



with(DEtools):

```
DEplot( diff(y(x),x)=y(x)-x,y(x),
x=-5..5, [[y(0)=1]],y=-5..5,stepsize=.05);
```

From the graph, we can see $y(3) = 4$.

5. Construct a table for y' , y'' and solution curve $y(x)$ shapes. Find and classify critical points Sketch phase portrait (phase line + direction field). Sketch all typical solution curves on the graph of direction field.

$$y' = 4y - y^2.$$

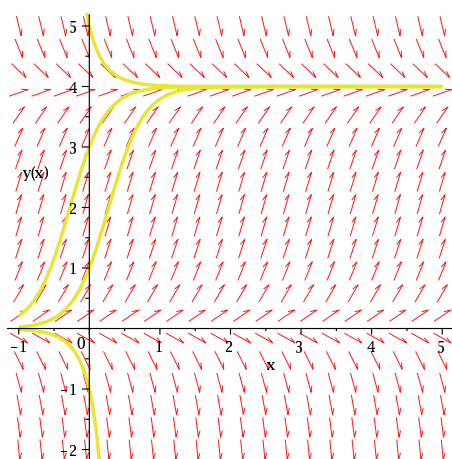
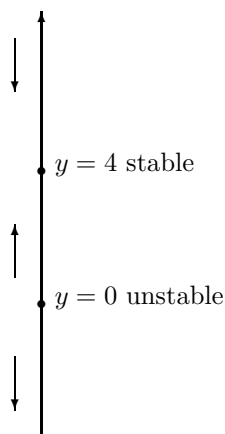
• **ans:**

$$\begin{aligned} y' &= 4y - y^2 = y(4 - y) \\ y'' &= 4y' - 2yy' = (4 - 2y)y' \end{aligned}$$

By $y' = 0$, we have critical points $y = 0$ and $y = 4$.

For $y'' = 0$, we have an additional point, $y = 2$.

t		0		2		4	
y	-		+		+		+
$4 - y$	+		+		+		-
$y' = y(4 - y)$	-		+		+		-
up/down	↘		↗		↗		↘
$(4 - 2y)$	+		+		-		-
$y'' = (4 - 2y)y'$	-		+		-		+
shape	()		()		()		()



```
with(DEtools):
DEplot( diff(y(x),x)=y(x)*4-y(x)^2,y(x),
x=-1..5, [[y(0)=1],[y(0)=3],[y(0)=5],
[y(0)=-1]],y=-2..5,
stepsize=.05);
```

6. Solve the equation and determine the interval in which the solution is defined.

$$y' = (1 - 2x)y^2, \quad y(0) = -1/6$$

- **ans:** When $y \neq 0$,

$$\frac{dy}{y^2} = (1 - 2x)dx$$

$$-\frac{1}{y} = x - x^2 + C$$

By the initial condition

$$6 = C$$

$$-\frac{1}{y} = x - x^2 + 6$$

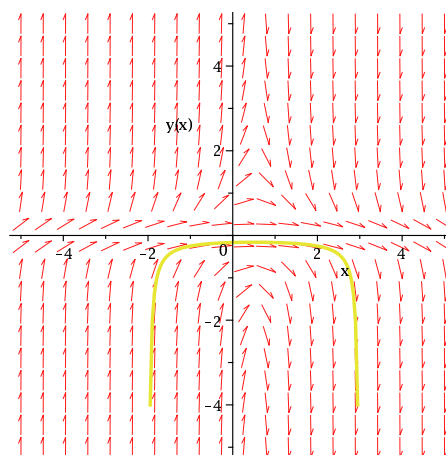
$$y = \frac{1}{-6 - x + x^2}$$

$$-6 - x + x^2 = 0$$

$$(x - 3)(x + 2) = 0, \quad x = -2, 3$$

Interval for the solution to exist

$$(-2, 3)$$



```
with(DEtools):
eq:=diff(y(x),x)=(1-2*x)*y(x)^2;
DEplot( eq, y(x),
x=-5..5, [[y(0)=-1/6]],y=-5..5,stepsize=.05);
dsolve({eq,y(0)=-1/6}, y(x));
```

7. Solve the IVP. Determine an interval for which the initial value problem is certain to have a unique solution.

$$y' = \frac{2t}{2 - 2y}, \quad y(3) = 2.$$

- **ans:** We solve the nonlinear equation first, then locate point t at which $y' = \pm\infty$.

$$(2 - 2y)dy = 2tdt$$

$$\int (2 - 2y)dy = \int (2t)dt$$

$$2y - y^2 = t^2 + C$$

$$y(3) = 2 \Rightarrow 4 - 4 = 9 + C, \quad C = -9$$

$$2y - y^2 = t^2 - 9$$

By the DE $y' = \pm\infty$ only when

$$2 - 2y = 0 \Rightarrow y = 1$$

When $y = 1$, by above solution

$$\begin{aligned} 2 - 1 &= t^2 - 9 \\ t^2 &= 10 \\ t &= \pm\sqrt{10} \end{aligned}$$

So the solution exists for $(-\infty, -\sqrt{10})$, $(-\sqrt{10}, \sqrt{10})$, $(\sqrt{10}, \infty)$. Since $y(3) = 2$, the answer is $(-\sqrt{10}, \sqrt{10})$.

- 2.54 8. Solve IVP. Find the largest interval over which the solution is defined.

$$y \frac{dx}{dy} - x = 2y^2, \quad y(1) = 5$$

• **ans:** Note that we solve $x(y)$ as a function of y . This way, we get a first order linear equation.

$$\begin{aligned} \frac{dx}{dy} - \frac{1}{y}x &= 2y \\ \mu &= e^{\int p} = e^{-\int 1/y} = e^{-\ln y} = \frac{1}{y} \\ x &= \frac{1}{\mu} \int \mu q = y \int 2dy \\ &= y(2y + C) \end{aligned}$$

By $y(1) = 5$, we replace y by 5 and x by 1:

$$\begin{aligned} 1 &= 5(10 + C), \quad C = -\frac{49}{5}, \\ x &= 2y^2 - \frac{49}{5}y \end{aligned}$$

Here when taking $x(y)$ as a function of y , the solution exists for all y , $(-\infty, \infty)$.

- End of solution. -

But if we take y as a function of x , then we can solve the equation to get

$$\begin{aligned} 2y^2 - \frac{49}{5}y - x &= 0 \\ y &= \frac{49/5 \pm \sqrt{(49/5)^2 + 8x}}{4} \end{aligned}$$

For the value inside a square root, we need

$$\begin{aligned} (49/5)^2 + 8x &\geq 0, \\ x &\geq -\frac{49^2}{5^2(8)} = -\frac{2401}{200} \\ &= -12.005 \end{aligned}$$

So the largest interval (containing $y(1)$) is $(-12.005, \infty)$.

There is another way. When the solution $y = y(x)$ breaks down, $y' = \infty$. So, by

$$x = 2y^2 - \frac{49}{5}y$$

we take derivatives

$$1 = (4y - \frac{49}{5})y'$$

As $y' = \infty$

$$0 = 4y - \frac{49}{5}, \quad y = \frac{49}{20}$$

Plug this y back into $x(y)$:

$$\begin{aligned} x &= 2y^2 - \frac{49}{5}y \\ &= 2\frac{49^2}{20^2} - \frac{49^2}{20(5)} \\ &= \frac{49^2}{20(5)}\left(\frac{1}{2} - 1\right) = -\frac{49^2}{200} \\ &= -12.005 \end{aligned}$$

So the largest interval (containing $y(1)$) is $(-12.005, \infty)$.

- 2.55 9. Solve

$$y' + y = \begin{cases} 1 & x < 1 \\ -1 & x \geq 1 \end{cases} \quad y(0) = 1$$

• **ans:** The solution is done in two steps.

First, we solve

$$y' + y = 1, \quad y(0) = 1$$

$$\begin{aligned} \mu &= e^{\int p} = e^{\int 1} = e^x \\ y &= \frac{1}{\mu} \int \mu q = e^{-x} \int e^x \\ &= e^{-x}(e^x + C) \end{aligned}$$

By $y(0) = 1$,

$$\begin{aligned} 1 &= 1 + C, \quad C = 0 \\ y &= e^{-x}e^x = 1 \end{aligned}$$

Then we evaluate y at $x = 1$, to get $y = 1$ (usually we get e or e^2 or like that.)

In the second step, we solve the differential equation again, but with new right hand side function and new initial condition:

$$y' + y = -1, \quad y(1) = 1$$

$$\begin{aligned} \mu &= e^{\int p} = e^{\int 1} = e^x \\ y &= \frac{1}{\mu} \int \mu q = e^{-x} \int -e^x \\ &= e^{-x}(-e^x + C) \end{aligned}$$

By $y(1) = 1$,

$$\begin{aligned} 1 &= -1 + e^{-1}C, \quad C = 2e \\ y &= -1 + 2ee^{-x} \end{aligned}$$

Hence, combine them together, we get

$$y = \begin{cases} 1 & x < 1 \\ -1 + 2ee^{-x} & x \geq 1 \end{cases}$$

10. Determine if the differential equation is exact, and solve it if it is.

$$(2y - \frac{1}{x} + \cos 3x) \frac{dy}{dx} + \frac{y}{x^2} - 4x^3 + 3y \sin 3x = 0$$

- **ans:** Rewrite the equation in the form $Mdx + Ndy = 0$:

$$(\frac{y}{x^2} - 4x^3 + 3y \sin 3x)dx + (2y - \frac{1}{x} + \cos 3x)dy = 0$$

Check:

$$M_y = \frac{1}{x^2} + 3 \sin 3x$$

$$N_x = \frac{1}{x^2} - 3 \sin 3x$$

It is not an exact equation. We do not know how to solve it.

11. Solve the IVP:

$$(\frac{1}{1+y^2} + \cos x - 2xy) \frac{dy}{dx} = y(y + \sin x), \quad y(0) = 1$$

- **ans:** Rewrite the equation in the form $Mdx + Ndy = 0$:

$$(y^2 + y \sin x)dx + (-\frac{1}{1+y^2} - \cos x + 2xy)dy = 0$$

Check (no need to check if we do the next step and find a solution):

$$M_y = 2y + \sin x$$

$$N_x = \sin x + 2y$$

It is an exact equation.

$$F = \int Mdx = \int (y^2 + y \sin x)dx$$

$$= xy^2 - y \cos x + c(y)$$

$$F_y = N$$

$$2xy - \cos x + c'(y) = -\frac{1}{1+y^2} - \cos x + 2xy$$

$$c'(y) = -\frac{1}{1+y^2}$$

$$c(y) = \int -\frac{1}{1+y^2} dy = -\tan^{-1} y$$

$$F = xy^2 - y \cos x - \tan^{-1} y$$

The general solution is $F = c$:

$$xy^2 - y \cos x - \tan^{-1} y = c$$

By $y(0) = 1$,

$$0 - 1 - \tan^{-1} 1 = c$$

$$c = -1 - \frac{\pi}{4}$$

So

$$xy^2 - y \cos x - \tan^{-1} y = -\frac{\pi}{4} - 1$$

12. Solve the equation of $y' = f(y/x)$ form:

$$\frac{dy}{dx} = \frac{x + 3y}{3x + y}$$

- **ans:** Let $y = ux$

$$u + xu' = y' = \frac{x + 3xu}{3x + xu}$$

$$u + xu' = \frac{1 + 3u}{3 + u}$$

Separable

$$x \frac{du}{dx} = \frac{1 + 3u}{3 + u} - u$$

$$\frac{du}{\frac{1+3u}{3+u} - u} = \frac{dx}{x}$$

$$\int \frac{(3+u)du}{1-u^2} = \int \frac{dx}{x}$$

We use partial fractions:

$$\frac{3+u}{1-u^2} = \frac{A}{1-u} + \frac{B}{1+u}$$

$$3+u = A(1+u) + B(1-u)$$

$$(u=1) \Rightarrow A=2$$

$$(u=-1) \Rightarrow B=1$$

So the above equation is

$$\int \frac{2du}{1-u} + \frac{du}{1+u} = \ln x + C$$

$$-\ln |1-u|^2 + \ln |1+u| = \ln x + C_1$$

$$\frac{|1+u|}{(1-u)^2} = Cx$$

$u = y/x$:

$$\frac{|1+y/x|}{(1-y/x)^2} = Cx$$

$$\frac{|x^2 + xy|}{(x-y)^2} = Cx$$

$$C(x-y)^2 = (x+y)$$

13. Solve the equation of $y' = f(Ax + By + C)$ form:

$$t_1 = 1.1$$

$$y' = \frac{1 - x - y}{x + y}$$

$$y_2 = y_1 + hf(t_1, y_1) = 0.3 + 0.1(3.8) = 0.68$$

• **ans:** We use the following substitution:

$$y' = f(Ax + By + C), \quad u = Ax + By + C$$

$$\text{error: } |y(t_i) - y_i|$$

Here,

$$\text{relative error: } \frac{|y(t_i) - y_i|}{|y(t_i)|}$$

$$\begin{aligned} u &= x + y \\ u' &= 1 + y' \\ &= 1 + \frac{1 - x - y}{x + y} = 1 + \frac{1 - u}{u} \\ u' &= \frac{1}{u} \end{aligned}$$

t_n	y_n	$y(t_n)$	abs error	rel err
1	0	0	0	0
1.1	0.3	0.3428	0.0428	0.1248
1.2	0.68	0.7836	0.1036	0.1322

Again, $h = 0.05$.

It is separable:

$$udu = dx$$

$$t_0 = 1, \quad h = 0.1, \quad y_0 = 0$$

$$\frac{1}{2}u^2 = x + c$$

$$y_1 = y_0 + hf(t_0, y_0) = 0 + 0.05(3) = 0.15$$

$$t_1 = 1.05$$

$$\frac{1}{2}(x + y)^2 = x + c$$

$$y_2 = y_1 + hf(t_1, y_1) = 0.15 + 0.05(3.4) = 0.32$$

14. Approximate $y(1.2)$ by the Euler method with $h = 0.1$ and $h = 0.05$ (using 4-decimal approximation):

$$y' = 2y + 2t + 1, \quad y(1) = 0$$

$$t_2 = 1.10$$

Compare the numerical solutions with exact solution listing errors and relative errors.

$$y_3 = 0.32 + 0.05(3.84) = 0.512$$

• **ans:** First we find the exact solution.

$$t_3 = 1.15$$

$$y' - 2y = 2t + 1$$

$$\mu = e^{\int p} = e^{\int -2dt} = e^{-2t}$$

$$y_3 = 0.512 + 0.05(4.324) = 0.7282$$

$$y = \frac{1}{\mu} \int \mu q = e^{2t} \int e^{-2t}(2t + 1)$$

$$= e^{2t}(e^{-2t}(-t - 1) + C) = -t - 1 + Ce^{2t}$$

$$y(1) = 0$$

$$0 = -1 - 1 + Ce^2$$

$$C = 2e^{-2}$$

$$y = -t - 1 + 2e^{2t-2}$$

$$y(1.2) = 0.7836$$

Now we apply Euler method with $h = 0.1 - 2$ steps for $y(1.2)$.

$$t_i = t_0 + ih, \quad y_{i+1} = y_i + hf(t_i, y_i)$$

$$t_0 = 1, \quad h = 0.1, \quad y_0 = 0$$

$$y_1 = y_0 + hf(t_0, y_0) = 0 + 0.1(3.000) = 0.3$$

t_n	y_n	$y(t_n)$	abs error	rel err
1	0	0	0	0
1.05	0.15	0.1603	0.0103	0.0644
1.1	0.32	0.3428	0.0228	0.0665
1.15	0.512	0.5497	0.0377	0.0686
1.2	0.7282	0.7836	0.0554	0.0707

15. A 4-lb roast, initially at 50°F, is placed in a 375°F oven at 5:00PM. After 75 minutes it is found that the temperature $T(t)$ of the roast is 125°F. When will the roast be 150°F (medium rare)?

• **ans:** When $t = 0$, it is 5pm. We have the model, the time rate of change of the temperature of a body immersed in a medium of constant temperature A is proportional to the difference $A - T$:

$$\frac{dT}{dt} = k(A - T)$$

What we know:

$$\begin{aligned} T(0) &= 50 \\ T(75) &= 125 \\ A &= 375. \end{aligned}$$

$$\frac{dT}{dt} = k(375 - T)$$

$$\int \frac{dT}{375 - T} = \int k dt$$

$$-\ln(375 - T) = kt + C, \quad T = 375 - Be^{-kt}$$

$$T(0) = 50$$

$$T = 375 - 325e^{-kt}$$

$$t = 75,$$

$$125 = 375 - 325e^{-k75},$$

$$k = -\frac{1}{75} \ln \frac{250}{325} = 0.0035$$

$$150 = 375 - 325e^{-0.0035t},$$

$$t = -\frac{\ln(225/325)}{0.0035} = 105.$$

We can skip the step of finding k :

$$125 = 375 - 325e^{-k75},$$

$$150 = 375 - 325e^{-kt},$$

$$\left(\frac{125 - 375}{-325}\right)^t = \left(\frac{150 - 375}{-325}\right)^{75}$$

$$t = \frac{\ln\left(\frac{150-375}{-325}\right)^{75}}{\ln\left(\frac{125-375}{-325}\right)} = 105.12.$$

16. A tank contains 100 liters of salt water with a concentration of 2 g/liter. Pure water flows in at a rate of 2 liters/min, the well-stirred solution flowing out at the same rate. Find the time when the tank reaches 1 g/liter salt concentration.

• **ans:** Volume 100, mass $d(t)$.

$$d' = -\frac{d}{100}2, \quad d(0) = 2 \cdot 100$$

$$d = 200e^{-t/50}$$

$$d(t) = 1 \cdot 100, \quad t = 50 \ln 2 = 34.6$$

17. The time rate of change of an alligator population P in a swamp is proportional to the square of P . The swamp contained a dozen alligators in 1988, two dozen in 1998. When will there be four dozen alligators in the swamp? What happens there after?

• **ans:**

$$P' = kP^2, \quad -\frac{1}{P} = kt - C$$

$$P = \frac{1}{C - kt}$$

$$P(0) = 12, \quad C = \frac{1}{12}, \quad P = \frac{12}{1 - 12kt}$$

$$P(10) = 24, \quad k = \frac{1}{240}, \quad P = \frac{240}{20 - t}$$

$$P = 48, \quad t = 15$$

$$t \rightarrow 20, \quad P \rightarrow \infty$$

18. Find the linear dependence by both finding nonzero solutions and by Wronskian:

$$y_1 = 1 + x^2, \quad y_2 = x^2 - x, \quad y_3 = x^2 + x$$

• **ans:** (1) By finding non zero solution:

$$c_1y_1 + c_2y_2 + c_3y_3 = 0$$

$$c_1(1 + x^2) + c_2(x^2 - x) + c_3(x^2 + x) = 0$$

Comparing coefficients:

$$x^2: \quad c_1 + c_2 + c_3 = 0$$

$$x: \quad -c_2 + c_3 = 0$$

$$1: \quad c_1 = 0$$

We have unique solution

$$c_1 = c_2 = c_3 = 0$$

Therefore, the three functions are linear independent. (If we can find one set of non-zero solutions, they are linearly dependent.)

(2) By Wronskian:

$$\begin{aligned} W &= \begin{vmatrix} 1 + x^2 & x^2 - x & x^2 + x \\ 2x & 2x - 1 & 2x + 1 \\ 2 & 2 & 2 \end{vmatrix} \\ &= (1 + x^2) \begin{vmatrix} 2x - 1 & 2x + 1 \\ 2 & 2 \end{vmatrix} - (x^2 - x) \begin{vmatrix} 2x & 2x + 1 \\ 2 & 2 \end{vmatrix} \\ &\quad + (x^2 + x) \begin{vmatrix} 2x & 2x - 1 \\ 2 & 2 \end{vmatrix} \\ &= (1 + x^2)(-4) - (x^2 - x)(-2) + (x^2 + x)(2) \\ &= -4 \neq 0 \end{aligned}$$

So linearly independent.

19. Given one solution, find another linear independent solution and the general solution:

$$x^2 y'' - y = 0 \quad y_1 = x^3$$

Here we find y_2 by $y_2 = u(x)y_1$ and get

$$y_2 = y_1 \int \frac{e^{\int -P}}{y_1^2}$$

- **ans:** Rewrite the equation in the standard form

$$y'' - \frac{1}{x^2}y = 0$$

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -P}}{y_1^2} \\ &= x^3 \int \frac{e^{\int 0}}{x^6} dx \\ &= x^3 \int x^{-6} dx = x^3 \left(-\frac{1}{5}x^{-5} + C \right) \\ &= -\frac{1}{5}x^{-2} + Cy_1 \end{aligned}$$

We can let $C = 0$ above as it is included in the general solution below:

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 x^3 + c_2' x^{-2} \end{aligned}$$

20. Find the roots of characteristic equation and the general solution:

$$(1) 4y'' - 4y' + y = 0$$

$$(2) (D-1)^2(D^2-1)^2((D-2)^2-9)((D-1)^2+4)^2 y = 0$$

- **ans:** (1) Characteristic equation and roots

$$\begin{aligned} 4r^2 - 4r + 1 &= 0, \quad (2r-1)^2 = 0, \\ r &= \frac{1}{2}, \frac{1}{2} \end{aligned}$$

Repeated roots.

The general solution is

$$y = Ae^{x/2} + Bxe^{x/2}$$

- (2) Characteristic equation and roots

$$\begin{aligned} (r-1)^2(D^2-1)^2((D-2)^2-9)((D-1)^2+4)^2 &= 0, \\ r = 1, 1; \quad 1, -1, 1, -1; \quad -1, 5; \quad 1 \pm 2i, 1 \pm 2i \end{aligned}$$

The general solution is

$$\begin{aligned} y &= (A + Bx + Cx^2 + Dx^3)e^x + \\ &\quad + (F + Gx + Hx^2)e^{-x} + Ie^{5x} \\ &\quad + e^x((J + Kx) \cos 2x + (L + Mx) \sin 2x) \end{aligned}$$