M351 H12 (S. Zhang) 8.12.

1. 8.12: 9-17, 39-40

1. (8.12:10) Factor the matrix A into a product XDX^{-1}, where D is diagonal.

$$(A = \begin{pmatrix} 1 & 2 \\ -1/2 & 1 \end{pmatrix}.$$

- **ans:** Find eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -1/2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = 0.$$

$$(1-\lambda)^2 = -1,\ (1-\lambda) = \pm i,\quad \lambda = 1 \pm i$$

For $\lambda = 1+i$, solve $(A - \lambda I)x = 0$:

$$\begin{pmatrix} -i & 2 \\ * & * \end{pmatrix},\ x = c\begin{pmatrix} 2 \\ i \end{pmatrix}$$

(Note: we do not need to write and eliminate the second row. We can choose x_1 and x_2 to be the coefficient of the other with one extra minus sign.)

For $\lambda = 1-i$, solve $(A - \lambda I)x = 0$:

$$\begin{pmatrix} i & 2 \\ * & * \end{pmatrix},\ x = c\begin{pmatrix} 2 \\ -i \end{pmatrix}$$

$$X = \begin{pmatrix} 2 & 2 \\ i & -i \end{pmatrix}$$
$$X^{-1} = \frac{1}{4}\begin{pmatrix} i & 1 \\ -1 & -2 \end{pmatrix}$$
$$A = XDX^{-1}$$
$$= X\begin{pmatrix} 1+i & \\ & 1-i \end{pmatrix}X^{-1}$$

2. (8.12:15) Factor the matrix A into a product XDX^{-1}, where D is diagonal.

$$(A = \begin{pmatrix} 1 & 3 & -1 \\ & 2 & 4 \\ & & 1 \end{pmatrix}.$$

- **ans:** Find eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & -1 \\ & 2-\lambda & 4 \\ & & 1-\lambda \end{vmatrix}$$
$$= (1-\lambda)(2-\lambda)(1-\lambda) = 0.$$

$$\lambda = 1, 1, 2$$

For $\lambda = 1, 2$, solve $(A - \lambda I)x = 0$ (need two linearly independent eigenvectors):

$$\begin{pmatrix} 0 & 3 & -1 \\ & 1 & 4 \\ & & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 3 & -1 \\ & & 4+1/3 \\ & & 0 \end{pmatrix},\ x = c\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(Note: Only x_1 is free, which does not appear in $Ax = 0$.)

For $\lambda = 2$, solve $(A - \lambda I)x = 0$:

$$\begin{pmatrix} -1 & 3 & -1 \\ & & 4 \\ & & -1 \end{pmatrix}\ x = c\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

(Note: Only x_2 is free.) Since we lack one linearly independent vector for $\lambda = 1, 1$, the matrix is defective and it is not diagonalizible.

3. (8.12:40) Factor the matrix A into a product XDX^{-1}, where D is diagonal. Then find A^{10} by the factorization.

$$A = \begin{pmatrix} 6 & -10 \\ 3 & -5 \end{pmatrix}$$

- **ans:** Find eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -10 \\ 3 & -5-\lambda \end{vmatrix} = (\lambda^2 - \lambda - 30) + 30 = 0.$$

$$\lambda = 0, 1$$

For $\lambda = 0$, solve $(A - \lambda I)x = 0$

$$\begin{pmatrix} 6 & -10 \\ 3 & -5 \end{pmatrix} \to \begin{pmatrix} 6 & -10 \\ & 0 \end{pmatrix},\ x = c\begin{pmatrix} 10 \\ 6 \end{pmatrix}$$

(Note: we can choose simpler eigenvectors. Here we simply choose x_1 and x_2 to be the coefficent of the other, with one sign reversed.)

For $\lambda = 1$, solve $(A - \lambda I)x = 0$:

$$\begin{pmatrix} 5 & -10 \\ 3 & -6 \end{pmatrix}\ x = c\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Match the eigenvectors with eigenvalues, we find the diagonalization of A:

$$X = \begin{pmatrix} 10 & 2 \\ 6 & 1 \end{pmatrix}$$
$$X^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 2 \\ 6 & -10 \end{pmatrix}$$
$$A = XDX^{-1} = X\begin{pmatrix} 0 & \\ & 1 \end{pmatrix}X^{-1}$$
$$A^{10} = XD^{10}X^{-1}$$
$$= X\begin{pmatrix} 0^{10} & 0 \\ 0 & 1^{10} \end{pmatrix}X^{-1}$$
$$= XDX^{-1} = A = \begin{pmatrix} 6 & -10 \\ 3 & -5 \end{pmatrix}$$