M351 H11 (S. Zhang) 8.7-10.

1. 8.7: 5-8

1. (8.7:6) Solve $Ax = b$ by Cramer's rule

$$A = \begin{pmatrix} 5 & 4 \\ 10 & -6 \end{pmatrix},\ b = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

- **ans:**

$$x_k = \frac{|B_k|}{\det A}$$

where B_k is obtained by replacing k-th column of A.

$$x = \frac{1}{\det A}\begin{pmatrix} \det B_1 \\ \det B_2 \end{pmatrix} = \frac{1}{\begin{vmatrix} 5 & 4 \\ 10 & -6 \end{vmatrix}}\begin{pmatrix} \begin{vmatrix} -1 & 4 \\ 5 & -6 \end{vmatrix} \\ \begin{vmatrix} 5 & -1 \\ 10 & 5 \end{vmatrix} \end{pmatrix} = \frac{1}{-70}\begin{pmatrix} -14 \\ 35 \end{pmatrix} = \begin{pmatrix} 1/5 \\ -1/2 \end{pmatrix}$$

2. (8.7:8) Solve $Ax = b$ by Cramer's rule

$$A = \begin{pmatrix} 1 & -1 & 6 \\ -1 & 2 & 4 \\ 2 & 3 & -1 \end{pmatrix},\ \mathbf{b} = \begin{pmatrix} -2 \\ 9 \\ 1/2 \end{pmatrix}$$

- **ans:**

$$x_k = \frac{|B_k|}{\det A}$$

where B_k is obtained by replacing k-th column of A.

$$x = \frac{1}{\det A}\begin{pmatrix} \det B_1 \\ \det B_2 \\ \det B_3 \end{pmatrix} = \frac{1}{\begin{vmatrix} 1 & -1 & 6 \\ -1 & 2 & 4 \\ 2 & 3 & -1 \end{vmatrix}}\begin{pmatrix} \begin{vmatrix} -2 & -1 & 6 \\ 9 & 2 & 4 \\ 1/2 & 3 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & -2 & 6 \\ -1 & 9 & 4 \\ 2 & 1/2 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 & -2 \\ -1 & 2 & 9 \\ 2 & 3 & 1/2 \end{vmatrix} \end{pmatrix} = \frac{1}{-63}\begin{pmatrix} 173 \\ -136 \\ -61/2 \end{pmatrix}$$

---

1. 8.8: 3-5, 8-11, 16-19, 21-22

1. (8.8:11) Find eigenvalues and eigenvectors.

$$\begin{pmatrix} -1 & 2 \\ -5 & 1 \end{pmatrix}$$

- **ans:**

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 2 \\ -5 & 1-\lambda \end{vmatrix},$$
$$\lambda = \pm 3i$$

$\lambda = 3i$

$$A - \lambda I = \begin{pmatrix} -1-3i & 2 \\ -5 & 1-3i \end{pmatrix} \to \begin{pmatrix} -1-3i & 2 \\ & 0 \end{pmatrix}$$

Choose the opposite coeffcients with one negative sign:

$$x = C\begin{pmatrix} 2 \\ 1+3i \end{pmatrix}$$

$\lambda = -3i$

$$A - \lambda I = \begin{pmatrix} -1+3i & 2 \\ -5 & 1+3i \end{pmatrix} \to \begin{pmatrix} -1+3i & 2 \\ & 0 \end{pmatrix}$$

Choose the opposite coeffcients with one negative sign:

$$x = C\begin{pmatrix} 2 \\ 1-3i \end{pmatrix}$$

Note that the two roots are conjugate, and the two eigenvectors are conjugate.

2. (8.8:18) Find eigenvalues and eigenvectors.

$$\begin{pmatrix} 1 & 6 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

- **ans:**

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 6 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix},$$
$$\lambda = 1, 1, 3$$

$\lambda = 1, 1$ (repeated roots)

$$A - \lambda I = \begin{pmatrix} 0 & 6 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

x_1 is free: (only 1 freedom, not two)

$$x = C\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(note, it is called a defficit eigenvalue, and the matrix cannot be diagonized.)

$\lambda = 3$

$$A - \lambda I = \begin{pmatrix} -2 & 6 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & -3 & 0 \\ & 1 & -1 \\ & & 0 \end{pmatrix}$$

x_3 is free:

$$x = C \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

---

1. 8.10: 5-7, 13-16, 21-22

1. (8.10:16) Find Orthogonal eig-matrices:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

- **ans:**

$$|A - \lambda I| \overset{\text{by row 1}}{=} (-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix}$$
$$+ \begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix}$$
$$= (-\lambda)(-1 - \lambda + \lambda^2) - (-1 - \lambda) + (lambda)$$
$$= -\lambda^3 + \lambda^2 + 3\lambda + 1$$

The only possible rational roots are ± 1. Trying both, we find one root $\lambda = -1$.

By doing a division, we get

$$|A - \lambda I| = -(\lambda + 1)(\lambda^2 - 2\lambda - 1)$$

$$\lambda = -1, 1 \pm \sqrt{2}$$

$\lambda = -1$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} K = 0, \ K_1 = C \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\lambda = 1 + \sqrt{2}$

$$\begin{pmatrix} -1-\sqrt{2} & 1 & 1 \\ 1 & -\sqrt{2} & 1 \\ 1 & 1 & -1-\sqrt{2} \end{pmatrix} K = 0$$

From symmetry, we let $x_1 = x_3 = 1$ then by first equation $x_2 = \sqrt{2}$ (we can check that they satisfy all three equations. We cheated a little here as we know we must find one nonzero solution, and we can choose that one constant smartly.)

$$K_2 = C \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$\lambda = 1 - \sqrt{2}$

$$\begin{pmatrix} -1+\sqrt{2} & 1 & 1 \\ 1 & +\sqrt{2} & 1 \\ 1 & 1 & -1+\sqrt{2} \end{pmatrix} K = 0$$

From symmetry, we let $x_1 = x_3 = 1$ then by first equation $x_2 = -\sqrt{2}$

$$K_2 = C \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

The eigenmatrix is

$$P = \begin{pmatrix} K_1 & K_2 & K_3 \end{pmatrix}$$

We check if its column is orthogonal: (After scaling its column vectors, we would get an orthogonal matrix.)

$$P^T P = \begin{pmatrix} 1 & 0 & -1 \\ 1 & \sqrt{2} & 1 \\ 1 & -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ -1 & 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & & \\ & 4 & \\ & & 4 \end{pmatrix}$$