M351 H10 (S. Zhang) 8.4-6.

1. 8.4: 5-11, 19-20, 25-28

1. (8.4:26) Find det A.

$$A = \begin{pmatrix} 2 & 1 & -2 & 1 \\ 0 & 5 & 0 & 4 \\ 1 & 6 & 1 & 0 \\ 5 & -1 & 1 & 1 \end{pmatrix}$$

- **ans:** Expansion by row 2, then by row 2 for the first 3×3 matrix and by row 1 for the second:

$$A = 5(-1)^{2+2} \begin{vmatrix} 2 & -2 & 1 \\ 1 & 1 & 0 \\ 5 & 1 & 1 \end{vmatrix} + 4(-1)^{2+4} \begin{vmatrix} 2 & 1 & -2 \\ 1 & 6 & 1 \\ 5 & -1 & 1 \end{vmatrix}$$
$$= 5(- \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix})$$
$$+ 4(2 \begin{vmatrix} 6 & 1 \\ -1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 6 \\ 5 & -1 \end{vmatrix})$$
$$= 5(3 - 3) + 4(14 + 4 + 62) = 320$$

2. 8.5: 7-14, 27-30, a1-a3

1. (8.5:a1) Find the determinant
(1) by cofactor expansions (no row/column operations)
(2) by row operations to an upper triangular matrix.
(3) by smart combinations of row/column op and expansion.

$$A = \begin{pmatrix} 2 & 0 & 4 & 0 \\ 5 & -10 & 0 & 5 \\ 0 & 2 & 4 & -2 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

- **ans:**

(1) Expansion by row 1

$$|A| = 2 \begin{vmatrix} -10 & 0 & 5 \\ 2 & 4 & -2 \\ 2 & 0 & 2 \end{vmatrix} + 4 \begin{vmatrix} 5 & -10 & 5 \\ 0 & 2 & -2 \\ 1 & 2 & 2 \end{vmatrix}$$

Then by row 1, and by row 2:

$$|A| = 2(-10 \begin{vmatrix} 4 & -2 \\ 0 & 2 \end{vmatrix} + 5 \begin{vmatrix} 2 & 4 \\ 2 & 0 \end{vmatrix})$$
$$+ 4(2 \begin{vmatrix} 5 & 5 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 5 & -10 \\ 1 & 2 \end{vmatrix})$$
$$= 2(-80 - 40) + 4(10 + 40) = -40$$

(2) row operations to an upper triangular matrix.

$$|A| \overset{\frac{1}{2}r_1}{=} 2 \begin{vmatrix} 1 & 0 & 2 & 0 \\ 5 & -10 & 0 & 5 \\ 0 & 2 & 4 & -2 \\ 1 & 2 & 0 & 2 \end{vmatrix}$$
$$\overset{\frac{1}{5}r_2}{=} 10 \begin{vmatrix} 1 & 0 & 2 & 0 \\ 1 & -2 & 0 & 1 \\ 0 & 2 & 4 & -2 \\ 1 & 2 & 0 & 2 \end{vmatrix}$$
$$\overset{\frac{1}{2}r_3}{=} 20 \begin{vmatrix} 1 & 0 & 2 & 0 \\ 1 & -2 & 0 & 1 \\ 0 & 1 & 2 & -1 \\ 1 & 2 & 0 & 2 \end{vmatrix}$$

$$|A| \overset{-r_1+r_2}{=} 20 \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & -2 & -2 & 1 \\ 0 & 1 & 2 & -1 \\ 1 & 2 & 0 & 2 \end{vmatrix}$$
$$\overset{-r_1+r_4}{=} 20 \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & -2 & -2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & -2 & 2 \end{vmatrix}$$
$$\overset{r_2 \leftrightarrow r_3}{=} -20 \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & -2 & -2 & 1 \\ 0 & 2 & -2 & 2 \end{vmatrix}$$
$$\overset{2r_2+r_3}{=} -20 \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 2 & -2 & 2 \end{vmatrix}$$

$$|A| \overset{-2r_2+r_4}{=} -20 \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -6 & 4 \end{vmatrix}$$
$$\overset{3r_3+r_4}{=} -20 \begin{vmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$
$$= -20(1)(1)(2)(1) = -40$$

(3) Smart combination:

$$\begin{vmatrix} 2 & 0 & 4 & 0 \\ 5 & -10 & 0 & 5 \\ 0 & 2 & 4 & -2 \\ 1 & 2 & 0 & 2 \end{vmatrix} \overset{-2c_1+c_3}{=} \begin{vmatrix} 2 & 0 & 0 & 0 \\ 5 & -10 & -10 & 5 \\ 0 & 2 & 4 & -2 \\ 1 & 2 & -2 & 2 \end{vmatrix}$$

$$\overset{\text{row 1 expan}}{=} 2\begin{vmatrix} -10 & -10 & 5 \\ 2 & 4 & -2 \\ 2 & -2 & 2 \end{vmatrix}$$

$$\overset{2c_3+c_1}{=} 2\begin{vmatrix} 0 & -10 & 5 \\ -2 & 4 & -2 \\ 6 & -2 & 2 \end{vmatrix}$$

$$\overset{2c_3+c_2}{=} 2\begin{vmatrix} 0 & 0 & 5 \\ -2 & 0 & -2 \\ 6 & 2 & 2 \end{vmatrix}$$

$$\overset{\text{column 3 expan}}{=} 2(5)\begin{vmatrix} -2 & 0 \\ 6 & 2 \end{vmatrix}$$

$$= 10(-4) = -40$$

2. (8.5:a3) $|A_{4\times 4}| = -2$, $|B_{4\times 4}| = -3$, find

$$|2A|,\ |A^T B|,\ |-2B^T A|,\ |B^2|$$

- **ans:** Factor 2 out of each row:

$$\begin{aligned} |2A| &= 2^4|A| = -32 \\ |A^T B| &= |A^T||B| = |A||B| = 6 \\ |-2B^T A| &= |-2B^T||A| = (-2)^4|B||A| = 96 \\ |B^2| &= |B||B| = 9 \end{aligned}$$

3. 8.6: 3-4,9-12, 17-20, 24-27, 29-30,46-48, 54-57

1. (8.6:12) Find A^{-1} by cofactors

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 8 & 0 & 0 \end{pmatrix}$$

- **ans:** Just using the formula:

$$A^{-1} = \frac{C^T}{|A|}$$

$$= \frac{\begin{pmatrix} \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 8 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 8 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 8 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 8 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} \end{pmatrix}^T}{\begin{vmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 8 & 0 & 0 \end{vmatrix}}$$

$$= \frac{1}{16}\begin{pmatrix} 0 & 8 & 0 \\ 0 & 0 & 16 \\ 2 & 0 & 0 \end{pmatrix}^T$$

$$= \begin{pmatrix} 0 & 0 & 1/8 \\ 1/2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

2. (8.6:24) Find A^{-1} by row operations:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 8 \end{pmatrix}$$

- **ans:**

$$\begin{aligned} (A\ I) &= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \\ &\overset{-2r_2+r_1}{\longrightarrow} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 & -2 & \\ & 1 & \\ & & 1 \end{pmatrix} \\ &\overset{(1/8)r_3}{\longrightarrow} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & \\ & 1 & \\ & & 1/8 \end{pmatrix} \\ &\overset{5r_3+r_1,-4r_3+r_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 5/8 \\ & 1 & -4/8 \\ & & 1/8 \end{pmatrix} \end{aligned}$$

$$A^{-1} = \begin{pmatrix} 1 & -2 & 5/8 \\ & 1 & -1/2 \\ & & 1/8 \end{pmatrix}$$