M351 H6 (S. Zhang) 3.7-8.

1. 3.7: a1-a2,3-4,13-15

1. (3.7:14) Taylor series solution of IVP:

$$y'' + y^2 = 1$$
$$y(0) = 2, \quad y'(0) = 3$$

- **ans:**

$$y'' = -y^2 + 1 \qquad y''(0) = -2^2 + 1 = -3$$

$$y''' = -2yy' \qquad y'''(0) = -12$$

$$y^{(4)} = -2yy'' - 2(y')^2 \qquad y^{(4)}(0) = -6$$

$$y^{(5)} = -2yy''' - 6(y')y'' \qquad y^{(4)}(0) = 102$$

Taylor formula:

$$y = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + ...$$

$$y = 2 + 3x - \frac{3}{2}x^2 - 2x^3 - \frac{1}{4}x^4 + \frac{17}{20}x^5$$

2. (3.7:a1) Solve the equation by 3 methods. (A) As a constant coefficient equation, (B) By reduction of order method, type no-y, (C) By reduction of order method, type no-x.

$$3y'' - 2y' = 0$$

- **ans:** (A)

$$3r^2 - 2r = 0, \ r = 0, 2/3$$

$$y = y_H = c + de^{2x/3}$$

(B) $u = y'$

$$3u' - 2u = 0$$

$$3\frac{du}{u} = 2dx$$

$$3\ln u = 2x + C$$

$$u = ce^{2x/3}$$

$$y = \int u = c\int e^{2x/3}$$
$$= c(\frac{3}{2}e^{2x/3} + d) = c_1e^{2x/3} + c_2$$

(C) $u = y'$, but

$$y'' = u\frac{du}{dy}$$

$$3u\frac{du}{dy} - 2u = 0$$

Divided by u (may check additional solution $u = 0$ below.)

$$3du = 2dy$$

$$3u = 2y + c_1$$

$$3\frac{dy}{dx} = 2y + c_1$$

$$\int \frac{3dy}{2y + c_1} = \int dx$$

$$\frac{3}{2}\ln(2y + c_1) = x + c_2$$

$$2y + c_1 = e^{\frac{2x}{3} + \frac{2}{3}c_2}$$

$$y = -\frac{1}{2}c_1 + \frac{1}{2}e^{\frac{2}{3}c_2}e^{2x/3}$$

$$y = C_1 + C_2e^{2x/3}$$

3. (3.7:a2) Solve the equation by the methods of reduction of order: (A) type no-y, (B) type no-x.

$$y'' - 2(y')^2 = 2$$

- **ans:** (A) $u = y'$

$$u' - 2u^2 = 2$$

$$\frac{du}{1 + u^2} = 2dx$$

$$\tan^{-1} u = 2x + C$$

$$u = \tan(2x + C)$$

$$y = \int u = \int \frac{\sin(2x + C)}{\cos(2x + C)}dx$$
$$= -\frac{1}{2}\ln\sin(2x + C) + D$$

(B) $u = y'$, but

$$y'' = u\frac{du}{dy}$$

$$u\frac{du}{dy} - 2u^2 = 2$$

Divided by $1 + u^2$

$$\int \frac{udu}{u^2+1} = \int 2dy$$

$$\frac{1}{2}\ln(u^2+1) = 2y + c_1$$

$$u^2 + 1 = c^2 e^{4y}, \quad c > 0, \quad c^2 = e^{c_1}$$

$$u = \pm\sqrt{c^2 e^{4y} - 1}$$

$$\pm\frac{dy}{\sqrt{c^2 e^{4y} - 1}} = dx$$

$$\pm\int e^{-2y}\frac{dy}{\sqrt{c^2 - e^{-4y}}} = \int dx$$

let

$$v = \frac{1}{c}e^{-2y}, \; dv = -\frac{2e^{-2y}dy}{c}$$

$$\pm\int(-\frac{c}{2})(\frac{1}{c})\frac{dv}{\sqrt{1-v^2}} = x + d$$

$$\pm\frac{1}{2}\sin^{-1} v = x + d$$

$$\sin^{-1}(\frac{1}{c}e^{-2y}) = \pm 2x + D$$

$$e^{-2y} = c\sin(\pm 2x + D)$$

$$e^{-2y} = C_1 \sin(2x + D_1)$$

$$\begin{aligned} y &= -\frac{1}{2}\ln(C_1 \sin(2x + D_1)) \\ &= -\frac{1}{2}\ln\sin(2x + D_1) + D_2 \end{aligned}$$

1. 3.8: 1, 6, 21, 22, 29, 31, 33, 45, 49, 53

1. (3.8:21) A mass of 4 pounds is attached to a spring whose constant is 2 lb/ft. The medium offers a damping force that is numerically equal to the instantaneous velocity. The mass is initially released from a point 1 foot above the equilibrium position with a downward velocity of 8 ft/s.

   Determine the time at which the mass passes through the equilibrium position.

   Find the time at which the mass attains its extreme displacement from the equailibrium position. What is the position of mass at this instant..

   - **ans:** Change the mass unit:

   $$m = \frac{4}{32} = \frac{1}{8}\text{lb-sec}^2/\text{ft}$$

   The gravity cancels a stretch of spring. We have only two forces, spring force and resistance force:

   $$\begin{aligned} mx'' + \gamma x' + kx &= 0 \\ \frac{1}{8}x'' + x' + 2x &= 0 \\ \frac{1}{8}r^2 + r + 2 &= 0 \\ r = -4, -4 \end{aligned}$$

   $$\begin{aligned} x(t) &= e^{-4t}(A + Bt) \\ x(0) &= -1, \; -1 = A \\ x'(0) &= 8, \; 4 = B \\ x(t) &= e^{-4t}(-1 + 4t) \end{aligned}$$

   First time passing equilibrium point, $x(t) = 0$

   $$0 = -1 + 4, \quad t = 1/4$$

   $$\begin{aligned} x'(t) &= e^{-4t}(4 - 16t + 4) \\ x'(t) &= 0, \\ 0 &= 8 - 16t, \quad t = 1/2 \\ x(1/2) &= e^{-4/2}(-1 + 4/2) = e^{-2} = 0.135 \end{aligned}$$

2. (3.8:53) Find the charge on the capacitor in an LRC-series circuit when

   $$L = \frac{1}{2}h, \; R = 10\Omega, \; C = 0.01f, \; E = 150V$$

   with initial condition

   $$q(0) = 1C, \; i(0) = 0A.$$

   What is the charge on the capacitor after a long time?

   - **ans:**

   $$\begin{aligned} LQ'' + RQ' + C^{-1}Q &= E \\ \frac{1}{2}Q'' + 10Q' + 100Q &= 150 \end{aligned}$$

Find Q_H:

$$\frac{1}{2}r^2 + 10r + 100 = 0,\ r = -10 \pm 10i$$

$$Q_H = e^{-10t}(A\cos 10t + B\sin 10t)$$

Find Q_P:

$$Q_P = C \ \Rightarrow C = \frac{3}{2}$$

$$Q = Q_H + Q_P = e^{-10t}(A\cos 10t + B\sin 10t) + \frac{3}{2}$$

By

$$Q(0) = 1,\ Q'(0) = 0$$

$$Q(t) = -\frac{1}{2}e^{-10t}(\cos 10t + \sin 10t) + \frac{3}{2}$$

When $t \to \infty$, $e^{-10t} \to 0$

$$Q(t) \to \frac{3}{2}C$$