M351 H5 (S. Zhang) 3.4-6.

1. 3.4: 7-8, 12-14, 17, a1-a2

20.99 1. (3.4:12) Solve

20.31

$$y'' - 16y = 2e^{4x}$$

- **ans:**

For y_H, char equation

$$r^{-}16 = 0$$

Roots are

$$r = \pm 4$$

$$y_H = c_1 e^{4x} + c_2 e^{-4x}$$

Because 4 is a root, we have <u>an extra</u> x in y_p:

$$y_p = x(Ae^{4x})$$

Plug it into equation, we fond $A = 1/4$.

$$y = y_H + y_p = c_1 e^{4x} + c_2 e^{-4x} + \frac{1}{4} x e^{4x}$$

2. (3.4:a1) Find the general solution of the <u>homogeneous</u>

20.41 equation y_H and <u>write down</u> the form for undetermined coefficients for y_p. (Do not solve the coefficients for y_p.)

$$\begin{aligned}(D^2-1)(D-1)(D^2+4)^2 y =& e^{-x} + 2e^x \sin 2x \\ & - 3x + x\cos 2x.\end{aligned}$$

- **ans:**

$$r = 1, -1, 1, \pm 2i, \pm 2i$$

$$\begin{aligned}y_c = y_H =& e^x(A + Bx) + e^x(E) \\ &+ ((G + Hx)\cos 2x + (I + Kx)\sin 2x. \\ y_p =& e^{-x}(A_1)x + e^x(B_1 \cos 2x + B_2 \sin 2x) + (C_1 x + C_2) \\ &+ ((D_1 x + D_2)\cos 2x + (E_1 x + E_2)\sin 2x)x^2\end{aligned}$$

3. (3.4:a1) Find the general solution of the <u>homogeneous</u>

20.42 equation y_c and <u>write down</u> the form for undetermined coefficients for y_p. (Do not solve for y_p.)

$$(D^2-9)(D+3)(D^2+4)y = e^x \sin x - e^{-3x} + \cos 2x.$$

- **ans:**

$$r = 3, -3, -3, \pm 2i$$

$$\begin{aligned}y_c = y_H =& e^{-3x}(A + Bx) + e^{3x}(E) + ((G)\cos 2x + (I)\sin 2x. \\ y_p =& e^x(A\cos x + B\sin x) + (C)x^2 e^{-3x} \\ &+ x(E\cos 2x + F\sin 2x)\end{aligned}$$

---

1. 3.5: 1, 2, 9-12, 25

22.99 1. (3.5:2) Solve

22.15

$$y'' + y = \tan x$$

- **ans:** For $y_c = y_H$,

$$r^2 + 1 = 0, \ r = \pm i$$

$$y_H = c_1 \cos x + c_2 \sin x$$

For y_p by VP, (note that the method of undetermined coefficients won't work here as $\tan x$ is not one of those special functions),

$$y_p = u_1 y_1 + u_2 y_2$$

satisfies eqautions:

$$\begin{aligned}u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= f\end{aligned}$$

Here we have

$$\begin{aligned}u_1' \cos x + u_2' \sin x &= 0 \\ -u_1' \sin x + u_2' \cos x &= \tan x\end{aligned}$$

$(equation1) * \cos x - (equation2) * \sin x$:

$$\begin{aligned}u_1' &= -\tan x \sin x \\ u_1 &= \int -\tan x \sin x = \int -\frac{\sin^2 x}{\cos x} \\ &= \int \frac{\cos^2 x - 1}{\cos x} = \int (\cos x - \sec x) \\ &= \int \cos x dx - \int \frac{\sec x(\sec x + \tan x)}{\sec x(\sec x + \tan x)} dx \\ &= \sin x - \ln(\sec x + \tan x)\end{aligned}$$

$(equation1) * \sin x - (equation2) * \cos x$:

$$\begin{aligned}u_2' &= \tan x \cos x = \sin x \\ u_2 &= \int \sin x dx = -\cos x\end{aligned}$$

The general solution:

$$\begin{aligned}y = y_H + y_P =& c_1 \cos x + c_2 \sin x \\ &+ (\sin x + \ln(\sec x + \tan x))\cos x \\ &+ (-\cos x)\sin x \\ =& c_1 \cos x + c_2 \sin x - \cos x \ln(\sec x + \tan x)\end{aligned}$$

Note that we can use the following formula to solve the two linear equations above:

$$\begin{aligned}u_1' &= \frac{-y_2 f}{W(y_1, y_2)} \\ u_2 &= \frac{y_1 f}{W(y_1, y_2)}\end{aligned}$$

Here

$$W(\cos x, \sin x) = 1$$

$$u_1 = \int \frac{-\sin x \tan x}{1} dx$$
$$u_2 = \int \frac{\cos x \tan x}{1} dx.$$

Then repeating above work.

2. (3.5:25) Solve

22.16

$$y''' + y' = \tan x$$

- **ans:** For $y_c = y_H$,

$$r^3 + r = 0, \ r = 0, \pm i$$

$$y_H = c_1 + c_2 \cos x + c_3 \sin x$$

For y_p by VP, (note that the method of undetermined coefficients won't work here as $\tan x$ is not one of those special functions),

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$$

satisfies eqautions:

$$\begin{aligned} u_1' y_1 + u_2' y_2 + u_3' y_3 &= 0 \\ u_1' y_1' + u_2' y_2' + u_3' y_3' &= 0 \\ u_1' y_1'' + u_2' y_2'' + u_3' y_3'' &= q \end{aligned}$$

Here we have

$$\begin{aligned} u_1' + u_2' \cos x + u_3' \sin x &= 0 \\ -u_2' \sin x + u_3' \cos x &= 0 \\ -u_2' \cos x - u_3' \sin x &= \tan x \end{aligned}$$

$(equation2) * \sin x + (equation3) * \cos x$:

$$\begin{aligned} -u_2' &= \tan x \cos x \\ u_2 &= \int -\tan x \cos x = -\int \sin x dx = \cos x \end{aligned}$$

$(equation2) * \cos x - (equation2) * \sin x$:

$$\begin{aligned} u_3' &= -\tan x \sin x \\ u_3 &= \int -\frac{\sin^2 x}{\cos x} \\ &= \int \frac{\cos^2 x - 1}{\cos x} = \int (\cos x - \sec x) \\ &= \int \cos x dx - \int \frac{\sec x(\sec x + \tan x)}{\sec x(\sec x + \tan x)} dx \\ &= \sin x - \ln(\sec x + \tan x) \end{aligned}$$

By $(equation1)$:

$$\begin{aligned} u_1' &= -u_2' \cos x - u_3' \sin x \\ &= \tan x \cos x \cos x + \tan x \sin x \sin x \\ &= \tan x \\ u_1 &= \int \tan x = -\ln \cos x \end{aligned}$$

The general solution:

$$\begin{aligned} y = y_H + y_P = & c_1 + c_2 \cos x + c_3 \sin x \\ & - \ln \cos x \\ & + (\cos x) \cos x \\ & + (\sin x + \ln(\sec x + \tan x)) \sin x \\ = & c_1' + c_2 \cos x + c_3 \sin x \\ & - \ln \cos x - \sin x \ln(\sec x + \tan x)) \end{aligned}$$

Note that we can use the formula in the book for solving the linear system of 3 equations above. But it takes more time to compute 4 determinants that way.

---

1. 3.6:4-9, 20-22

23.99

1. (3.6:8) Solve an Euler equation:

23.41

$$x^2 y'' + 3xy' - 4y = 0$$

- **ans:**

$$r(r-1) + 3r - 4 = 0$$

$$r = -1 \pm \sqrt{5}$$

$$y = c_1 x^{-1+\sqrt{5}}) + c_2 x^{-1-\sqrt{5}}$$

2. (3.6:9) Solve an Euler equation:

23.42

$$25x^2 y'' + 25xy' + y = 0$$

- **ans:**

$$25r(r-1) + 25r + 1 = 0$$

$$r = \pm \frac{1}{5} i$$

$$y = c_1 \sin(\frac{1}{5} \ln x)) + c_2 \sin(\frac{1}{5} \ln x)$$

3. (3.6:20) Solve an Euler equation:

23.43

$$2x^2 y'' + 5xy' + y = x^2 - x$$

- **ans:** For $y_H = y_c$:

$$2r(r-1) + 5r + 1 = 0$$

$$r = -1, -\frac{1}{2}$$

$$y = c_1\frac{1}{x} + c_2\frac{1}{\sqrt{x}}$$

For y_p, we have to use the method of variation of parameter as we have a non-constant coefficient equation, the method of undeteminend coefficients does not work.

However, we have to normalize the equation first!

$$y'' + \frac{5}{2x}y' + \frac{1}{2x^2}y = \frac{1}{2} - \frac{1}{2x}$$

$$y_p = u_1y_1 + u_2y_2$$

$$\begin{aligned} u_1'x^{-1} + u_2'x^{-1/2} &= 0 \\ -u_1'x^{-2} - \frac{1}{2}u_2'x^{-3/2} &= \frac{1}{2} - \frac{1}{2x} \end{aligned}$$

$(1/2)eq1/x + eq2$:

$$\begin{aligned} u_1'(\frac{1}{2} - 1)x^{-2} &= \frac{1}{2} - \frac{1}{2x} \\ u_1' &= (-1 + \frac{1}{x})x^2 \\ u_1 &= -\frac{1}{3}x^3 + \frac{1}{2}x^2 \end{aligned}$$

$eq1/x + eq2$:

$$\begin{aligned} u_2'(1 - \frac{1}{2})x^{-3/2} &= \frac{1}{2} - \frac{1}{2x} \\ u_2' &= (1 - \frac{1}{x})x^{3/2} \\ u_2 &= \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2} \end{aligned}$$

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 \\ &= (-\frac{1}{3}x^2 + \frac{1}{2}x) + (\frac{2}{5}x^2 - \frac{2}{3}x) \\ &= \frac{1}{15}x^2 - \frac{1}{6}x \end{aligned}$$

The general solution is

$$y = y_H + y_p = c_1\frac{1}{x} + c_2\frac{1}{\sqrt{x}} + \frac{1}{15}x^2 - \frac{1}{6}x$$