M351 H4 3.1-3 (S. Zhang) .

1. 3.1: 17-19, a1

11.99 1. (3.1:a1) Find the linear dependence by both finding
11.43 nonzero solutions and by Wrongskian:

$$y_1 = 1 + x^2,\ y_2 = x^2 - x,\ y_3 = x^2 + 3x + 4$$

- **ans:**

Method 1 (nonzero solutions)

$$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$$

By inspection:

$$\begin{aligned} 4y_1 + (-3)y_2 + (-1)y^3 &= 0 \\ 4(1+x^2) + (-3)(x^2 - x) + (-1)(x^2 + 3x + 4) &= 0 \end{aligned}$$

One set of nonzero solutions $\Rightarrow$ linearly dependent.

Here we can find other nonzero solutions. And we can choose some special values of x to find nonzero solutions, or compare coefficients, as follows

Method 1(b)

$$c_1(1+x^2) + c_2(x^2 - x) + c_3(x^2 + 3x + 4) = 0$$

$$\begin{aligned} x^2 &: c_1 + c_2 + c_3 = 0 \\ x &\ \ - c_2 + 3c_3 = 0 \\ 1 &\ \ c_1 + 4c_3 = 0 \end{aligned}$$

Let $c_3 = -1$. then $c_1 = 4$ and $c_2 = -3$. We get one set of nonzero solutions.

Method 2 (Wronskian)

$$W = \begin{vmatrix} 1+x^2 & x^2 - x & x^2 + 3x + 4 \\ 2x & 2x - 1 & 2x + 3 \\ 2 & 2 & 2 \end{vmatrix} = 0$$

Zero Wronskian $\Rightarrow$ linearly dependent.

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1. 3.2: 2-3, 10-11

12.99 2. (3.2:10) Given one solution, find another linear inde-
12.23 pendent solution and the general solution:

$$x^2 y'' + 2xy' - 6y = 0 \quad y_1 = x^2$$

Here we find y_2 by $y_2 = u(x) y_1$ and get

$$y_2 = y_1 \int \frac{e^{\int -P}}{y_1^2}$$

- **ans:** Rewrite the equation in the standard form

$$y'' + \frac{2}{x} y' - \frac{6}{x^2} y = 0$$

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -P}}{y_1^2} \\ &= x^2 \int \frac{e^{\int -2/x dx}}{x^4} dx \\ &= x^2 \int x^{-2} x^{-4} dx = x^2(-\frac{1}{5} x^{-5} + C) \\ &= -\frac{1}{5} x^{-3} \end{aligned}$$

We can let $C = 0$ above as it is included in the general solution below:

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 x^2 + c_2' x^{-3} \end{aligned}$$

Here also we absorb the constant $-1/5$ into the constant c_2.

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1. 3.3: 4-9, a1-a4

16.99 1. (3.3:a4) Find the roots of characteristic equation and
16.27 the general solution:

$$\begin{aligned} &(1)\ (D-2)^2(D^2+2)y = 0 \\ &(2)\ (D-2)^3(D^2-4)((D-2)^2+4)^2 y = 0 \end{aligned}$$

- **ans:** (1) Characteristic equation and roots

$$(r-2)^2(r^2+2) = 0,\ r = 2, 2, \pm\sqrt{2}i$$

The general solution is

$$y = Ae^{2x} + Bxe^{2x} + C\cos\sqrt{2}x + D\sin\sqrt{2}x$$

(2) Characteristic equation and roots

$$\begin{aligned} (D-2)^3(D^2-4)((D-2)^2+4)^2 &= 0, \\ r &= 2, 2, 2, 2, -2, 2 \pm 2i, 2 \pm 2i \end{aligned}$$

The general solution is

$$\begin{aligned} y =& (A + Bx + Cx^2 + Dx^3)e^{2x} + Fe^{-2x} \\ &+ e^{2x}((G + Hx)\cos 2x + (I + Jx)\sin 2x) \end{aligned}$$