M351 H2 2.2 2.3 2.4 2.5 (S. Zhang) .

1. (2.2:20) Solve

4.21

$$y' = \frac{xy + 2y - x - 2}{xy - 3y + x - 3}$$

- **ans:** We try to separate x and y.

$$\begin{aligned} y' &= \frac{xy + 2y - x - 2}{xy - 3y + x - 3} \\ &= \frac{(x+2)y - (x+2)}{(x-3)y + (x-3)} \\ &= \frac{(x+2)(y-1)}{(x-3)(y+1)} \\ \frac{y+1}{y-1}dy &= \frac{x+2}{x-3}dx \\ \frac{y-1+2}{y-1}dy &= \frac{x-3+5}{x-3}dx \\ \int (1 + \frac{2}{y-1})dy &= \int (1 + \frac{5}{x-3})dx \\ y + 2\ln|y-1| &= x + 5\ln|x-3| + C \end{aligned}$$

2. (2.2:a1) Solve the equation and determine the interval
4.51 in which the solution is defined.

$$y' = (1-2x)y^2, \ y(0) = -1/6$$

- **ans:** When $y \neq 0$,

$$\frac{dy}{y^2} = (1-2x)dx$$

$$-\frac{1}{y} = x - x^2 + C$$

By the initial condition

$$6 = C$$

$$-\frac{1}{y} = x - x^2 + 6$$

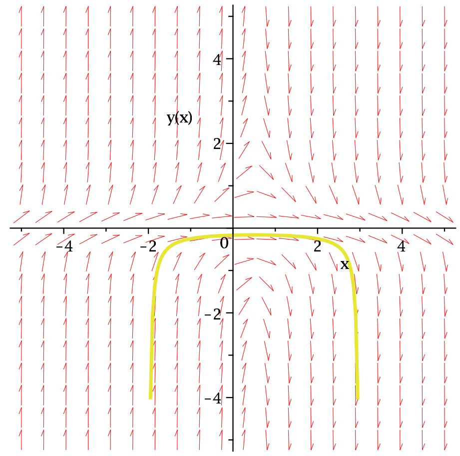
$$y = \frac{1}{-6 - x + x^2}$$

$$-6 - x + x^2 = 0$$

$$(x-3)(x+2) = 0, \ x = -2, \ 3$$

Interval for the solution to exist

$$(-2, 3)$$




```
with(DEtools):
eq:=diff(y(x),x)=(1-2*x)*y(x)^2;
DEplot( eq, y(x),
  x=-5..5,[[y(0)=-1/6]],y=-5..5,stepsize=.05);
dsolve({eq,y(0)=-1/6}, y(x));
```

3. (2.2:a2) Solve the IVP. Determine an interval for which
4.15 the initial value problem is certain to have a unique solution.

$$y' = \frac{2t}{2-2y}, \qquad y(3) = 2.$$

- **ans:** We solve the nonlinear equation first, then locate point t at which $y' = \pm\infty$.

$$\begin{aligned} (2-2y)dy &= 2tdt \\ \int (2-2y)dy &= \int (2t)dt \\ 2y - y^2 &= t^2 + C \\ y(3) = 2 \Rightarrow & \quad 4 - 4 = 9 + C, \ C = -9 \\ 2y - y^2 &= t^2 - 9 \end{aligned}$$

By the DE $y' = \pm\infty$ only when

$$2 - 2y = 0 \quad \Rightarrow \quad y = 1$$

When $y = 1$, by above solution

$$\begin{aligned} 2 - 1 &= t^2 - 9 \\ t^2 &= 10 \\ t &= \pm\sqrt{10} \end{aligned}$$

So the solution exists for $(-\infty, -\sqrt{10})$, $(-\sqrt{10}, \sqrt{10})$, $(\sqrt{10}, \infty)$. Since $y(3) = 2$, the answer is $(-\sqrt{10}, \sqrt{10})$.

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1. §2.3: 8-14, 25-26, 31-32

2.99

1. (2.3:25) Solve IVP. Find the largest interval over which
2.53 the solution is defined.

$$xy' + y = e^x, \ y(1) = 2$$

- **ans:**

$$
\begin{aligned}
xy' + y &= e^x, \\
y' + \frac{1}{x}y &= \frac{e^x}{x} \\
\mu &= e^{\int p} = e^{\int 1/x} = e^{\ln x} = x \\
y &= \frac{1}{\mu}\int \mu q = \frac{1}{x}\int e^x \\
&= \frac{1}{x}(e^x + C)
\end{aligned}
$$

By $y(1) = 2$

$$
\begin{aligned}
2 &= e + C \\
y &= \frac{1}{x}(e^x + 2 - e)
\end{aligned}
$$

Only when $x = 0$, y would be discontinuous. So the largest interval (containing $y(1)$) is $(0, \infty)$.

2. (2.3:26) Solve IVP. Find the largest interval over which
2.54 the solution is defined.

$$
y\frac{dx}{dy} - x = 2y^2, \ y(1) = 5
$$

- **ans:** Note that we solve $x(y)$ as a function of y. This way, we get a first order linear equation.

$$
\begin{aligned}
\frac{dx}{dy} - \frac{1}{y}x &= 2y \\
\mu &= e^{\int p} = e^{-\int 1/y} = e^{-\ln y} = \frac{1}{y} \\
x &= \frac{1}{\mu}\int \mu q = y\int 2dy \\
&= y(2y + C)
\end{aligned}
$$

By $y(1) = 5$, we replace y by 5 and x by 1:

$$
\begin{aligned}
1 &= 5(10 + C), \quad C = -\frac{49}{5}, \\
x &= 2y^2 - \frac{49}{5}y
\end{aligned}
$$

Here when taking $x(y)$ as a function of y, the solution exists for all y, $(-\infty, \infty)$.

– End of solution. –

But if we take y as a function of x, then we can solve the equation to get

$$
\begin{aligned}
2y^2 - \frac{49}{5}y - x &= 0 \\
y &= \frac{49/5 \pm \sqrt{(49/5)^2 + 8x}}{4}
\end{aligned}
$$

For the value inside a square root, we need

$$
\begin{aligned}
(49/5)^2 + 8x &\geq 0, \\
x &\geq -\frac{49^2}{5^2(8)} = -\frac{2401}{200} \\
&= -12.005
\end{aligned}
$$

So the largest interval (containing $y(1)$) is $(-12.005, \infty)$.

There is another way. When the solution $y = y(x)$ breaks down, $y' = \infty$. So, by

$$
x = 2y^2 - \frac{49}{5}y
$$

we take derivatives

$$
1 = (4y - \frac{49}{5})y'
$$

As $y' = \infty$

$$
0 = 4y - \frac{49}{5}, \quad y = \frac{49}{20}
$$

Plug this y back into $x(y)$:

$$
\begin{aligned}
x &= 2y^2 - \frac{49}{5}y \\
&= 2\frac{49^2}{20^2} - \frac{49^2}{20(5)} \\
&= \frac{49^2}{20(5)}(\frac{1}{2} - 1) = -\frac{49^2}{200} \\
&= -12.005
\end{aligned}
$$

So the largest interval (containing $y(1)$) is $(-12.005, \infty)$.

3. (2.3:32) Solve
2.55

$$
y' + y = \begin{cases} 1 & x < 1 \\ -1 & x \geq 1 \end{cases} \quad y(0) = 1
$$

- **ans:** The solution is done in two steps.

First, we solve

$$
y' + y = 1, \ y(0) = 1
$$

$$
\begin{aligned}
\mu &= e^{\int p} = e^{\int 1} = e^x \\
y &= \frac{1}{\mu}\int \mu q = e^{-x}\int e^x \\
&= e^{-x}(e^x + C)
\end{aligned}
$$

By $y(0) = 1$,

$$
\begin{aligned}
1 &= 1 + C, \ C = 0 \\
y &= e^{-x}e^x = 1
\end{aligned}
$$

Then we evaluate y at $x =$, to get $y = 1$ (usually we get e or e^2 or like that.)

In the second step, we solve the differential equation again, but with new right hand side function and new initial condition:

$$
y' + y = -1, \ y(1) = 1
$$

$$\mu = e^{\int p} = e^{\int 1} = e^x$$
$$y = \frac{1}{\mu}\int \mu q = e^{-x}\int -e^x$$
$$= e^{-x}(-e^x + C)$$

By $y(1) = 1$,

$$1 = -1 + e^{-1}C,\ C = 2e$$
$$y = -1 + 2ee^{-x}$$

Hence, combine them together, we get

$$y = \begin{cases} 1 & x < 1 \\ -1 + 2ee^{-x} & x \geq 1 \end{cases}$$

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1. 2.4: 6-12, 26-29

8.99

1. (2.4:6) Determine if the differential equation is exact,

8.42 and solve it if it is.

$$(2y - \frac{1}{x} + \cos 3x)\frac{dy}{dx} + \frac{y}{x^2} - 4x^3 + 3y\sin 3x = 0$$

- **ans:** Rewrite the equation in the form $Mdx + Ndy = 0$:

$$(\frac{y}{x^2} - 4x^3 + 3y\sin 3x)dx + (2y - \frac{1}{x} + \cos 3x)dy = 0$$

Check:

$$M_y = \frac{1}{x^2} + 3\sin 3x$$
$$N_x = \frac{1}{x^2} - 3\sin 3x$$

It is not an exact equation. We do not know how to solve it.

2. (2.4:26) Solve the IVP:

8.43

$$(\frac{1}{1+y^2} + \cos x - 2xy)\frac{dy}{dx} = y(y + \sin x),\ y(0) = 1$$

- **ans:** Rewrite the equation in the form $Mdx + Ndy = 0$:

$$(y^2 + y\sin x)dx + (-\frac{1}{1+y^2} - \cos x + 2xy)dy = 0$$

Check (no need to check if we do the next step and find a solution):

$$M_y = 2y + \sin x$$
$$N_x = \sin x + 2y$$

It is an exact equation.

$$F = \int M dx = \int (y^2 + y\sin x)dx$$
$$= xy^2 - y\cos x + c(y)$$
$$F_y = N$$
$$2xy - \cos x + c'(y) = -\frac{1}{1+y^2} - \cos x + 2xy$$
$$c'(y) = -\frac{1}{1+y^2}$$
$$c(y) = \int -\frac{1}{1+y^2}dy = -\tan^{-1} y$$
$$F = xy^2 - y\cos x - \tan^{-1} y$$

The general solution is $F = c$:

$$xy^2 - y\cos x - \tan^{-1} y = c$$

By $y(0) = 1$,

$$0 - 1 - \tan^{-1} 1 = c$$
$$c = -1 - \frac{\pi}{4}$$

So

$$xy^2 - y\cos x - \tan^{-1} y = -\frac{\pi}{4} - 1$$

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1. 2.5: 7-11, 24-26

9.99

1. (2.5:8) Solve the equation of $y' = f(y/x)$ form:

9.21

$$\frac{dy}{dx} = \frac{x + 3y}{3x + y}$$

- **ans:** Let $y = ux$

$$u + xu' = y' = \frac{x + 3xu}{3x + xu}$$
$$u + xu' = \frac{1 + 3u}{3 + u}$$

Separable

$$x\frac{du}{dx} = \frac{1+3u}{3+u} - u$$
$$\frac{du}{\frac{1+3u}{3+u} - u} = \frac{dx}{x}$$
$$\int \frac{(3+u)du}{1 - u^2} = \int \frac{dx}{x}$$

We use partial fractions:

$$\frac{3+u}{1-u^2} = \frac{A}{1-u} + \frac{B}{1+u}$$
$$3 + u = A(1+u) + B(1-u)$$
$$(u = 1) \Rightarrow\ A = 2$$
$$(u = -1) \Rightarrow\ B = 1$$

So the above equation is

$$\int \frac{2du}{1-u} + \frac{du}{1+u} = \ln x + C$$

$$-\ln|1-u|^2 + \ln|1+u| = \ln x + C_1$$

$$\frac{|1+u|}{(1-u)^2} = Cx$$

$u = y/x$:

$$\frac{|1+y/x|}{(1-y/x)^2} = Cx$$

$$\frac{|x^2+xy|}{(x-y)^2} = Cx$$

$$C(x-y)^2 = (x+y)$$

2. (2.5:24) Solve the equation of $y' = f(Ax + By + C)$
9.22 form:

$$y' = \frac{1-x-y}{x+y}$$

- **ans:** We use the following substitution:

$$y' = f(Ax+By+C),\ u = Ax+By+C$$

Here,

$$\begin{aligned} u &= x+y \\ u' &= 1+y' \\ &= 1 + \frac{1-x-y}{x+y} = 1 + \frac{1-u}{u} \\ u' &= \frac{1}{u} \end{aligned}$$

It is separable:

$$udu = dx$$

$$\frac{1}{2}u^2 = x + c$$

$$\frac{1}{2}(x+y)^2 = x + c$$