M351 H1 1.1, 1.2, 2.1 (S. Zhang) .

1. (1.1:25) Verify that the piecewise defined function

$$y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

is a solution of the differential equation

$$xy' - 2y = 0 \quad \text{on } (-\infty, \infty).$$

• **ans:** We have to check the function in three parts of x region.

(1) $x < 0$,

$$\begin{aligned} y &= -x^2 \\ y' &= -2x \\ x(-2x) - 2(-x^2) &= 0 \end{aligned}$$

(2) $x = 0$,

$$\begin{aligned} y(0) &= (x^2)_{x=0} = 0 \\ y'_+(0) &= \lim_{x\to 0^+} y'(x) = \lim_{x\to 0^+} 2x = 0 \\ y'_-(0) &= \lim_{x\to 0^-} y'(x) = \lim_{x\to 0^-} -2x = 0 \\ y'(0) &= y'_+(0) = y'_-(0) = 0 \end{aligned}$$

differential equation becomes to

$$\begin{aligned} xy' - 2y &= 0 \\ 0(0) - 2(0) &= 0 \end{aligned}$$

(3) $x > 0$,

$$\begin{aligned} y &= x^2 \\ y' &= 2x \\ x(2x) - 2(x^2) &= 0 \end{aligned}$$

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1. (1.2:11) Solve IVP with the general solution $y = C_1e^x + C_2e^{-x}$:

$$y'' - y = 0, y(0) = 1, y'(0) = 2$$

• **ans:**

$$\begin{aligned} y(0) = 1 &\Rightarrow 1 = C_1 + C_2 \\ y' = C_1e^x - C_2e^{-x} & \\ y'(0) = 2 &\Rightarrow 2 = C_1 - C_2 \\ &\Rightarrow C_1 = \frac{3}{2},\ C_2 = -\frac{1}{2} \\ y &= \frac{3}{2}e^x - \frac{1}{2}e^{-x} \end{aligned}$$

2. (1.2:18) Without solving the equation, find and plot the region in xy plane for the following equation such that there is a unique solution through each given initial point in this region

$$y' = \sqrt{xy}$$

• **ans:**

$$\begin{aligned} f = \sqrt{xy} &\Rightarrow xy \geq 0 \\ f_y = \frac{1}{2}\sqrt{\frac{x}{y}} &\Rightarrow y \neq 0, xy > 0 \end{aligned}$$

For any point (x_0, y_0) in the first and the third quadrant, excluding $y = 0$ line, we have locally a unique solution for IV $y(x_0) = y_0$.

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1. (2.1:22) Construct a table for y', y'' and solution curve $y(x)$ shapes. Find and classify critical points Sketch phase portrait (phase line + direction field). Sketchall typical solution curves on the graph of direction field.

$$y' = y^2 - y^3.$$

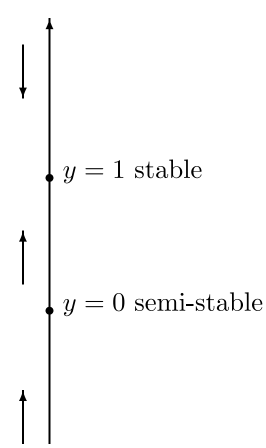
• **ans:**

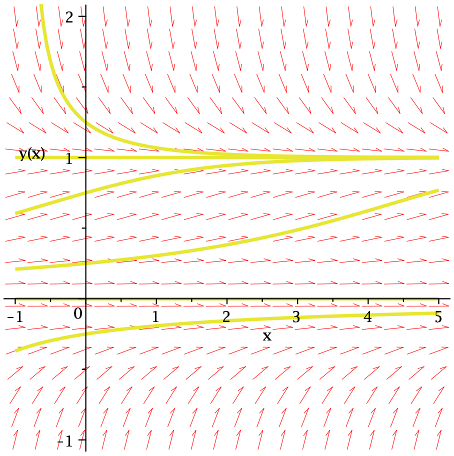
$$\begin{aligned} y' &= y^2 - y^3 = y^2(1-y) \\ y'' &= 2yy' - 3y^2y' = (2-3y)yy' = \end{aligned}$$

By $y' = 0$, we have critical points $y = 0$ and $y = 1$.

For $y'' = 0$, we have an additional point, $y = 2/3$.

t		0		$\frac{2}{3}$		1	
y^2	+		+		+		+
$1-y$	+		+		+		−
$y' = y^2(1-y)$	+		+		+		−
up/down	↗		↗		↗		↘
$(2-3y)$	+		+		−		−
y	−		+		+		+
$y'' = (2-3y)yy'$	−		+		−		+
shape	⌒		⌣		⌒		⌣






```
with(DEtools):
DEplot( diff(y(x),x)=y(x)^2-y(x)^3,y(x),
  x=-1..5,[[y(0)=0],[y(0)=1/4],[y(0)=3/4],
[y(0)=-1/4],[y(0)=1],[y(0)=5/4]],y=-1..2,
stepsize=.05);
```

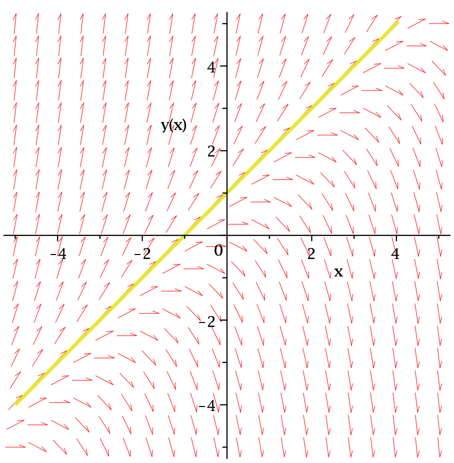
2. (2.1:a1) Make a table for y' and plot the direction field, including at least points $(0,1)$, $(1,1)$ and $(1,0)$. Then sketch the solution curve corresponding to the given initial condition. Finally, use this solution curve to estimate the desired value of the solution:

$$\frac{dy}{dx} = y - x; \quad y(0) = 1, \quad y(3) = ?$$

• **ans:**

(x,y)	$(0,1)$	$(1,1)$	$(1,0)$
$y' = y - x$	1	0	-1

We draw a little arrow at point $(0,1)$ with a slope $y' = 1$.




```
with(DEtools):
DEplot( diff(y(x),x)=y(x)-x,y(x),
  x=-5..5,[[y(0)=1]],y=-5..5,stepsize=.05);
```

From the graph, we can see $y(3) = 4$.

3. (2.1:a2) Construct a table for y', y'' and solution curve $y(x)$ shapes. Find and classify critical points Sketch phase portrait (phase line + direction field). Sketchall typical solution curves on the graph of direction field.

$$y' = 4y - y^2.$$

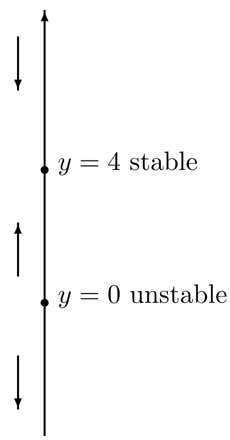
• **ans:**

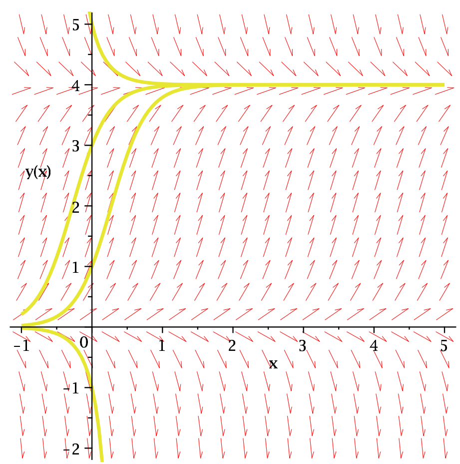
$$y' = 4y - y^2 = y(4-y)$$
$$y'' = 4y' - 2yy' = (4-2y)y'$$

By $y' = 0$, we have critical points $y = 0$ and $y = 4$.

For $y'' = 0$, we have an additional point, $y = 2$.

t		0		2		4	
y	$-$		$+$		$+$		$+$
$4-y$	$+$		$+$		$+$		$-$
$y' = y(4-y)$	$-$		$+$		$+$		$-$
up/down	$\searrow$		$\nearrow$		$\nearrow$		$\searrow$
$(4-2y)$	$+$		$+$		$-$		$-$
$y'' = (4-2y)y'$	$-$		$+$		$-$		$+$
shape	$\frown$		$\smile$		$\frown$		$\smile$






```
with(DEtools):
DEplot( diff(y(x),x)=y(x)*4-y(x)^2,y(x),
  x=-1..5,[[y(0)=1],[y(0)=3],[y(0)=5],
```

```
[y(0)=-1]],y=-2..5,
stepsize=.05);
```