M341 Study Guide for E 3 (S. Zhang) .

1. Find the row echelon form (okay to have non-one diagonals, Gauss elimination) and solve the system. Find the reduced row echelon form (Gauss-Jordan elimination) and solve the system.

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 3 \\ -2x_1 + x_2 + 0x_3 &= -3 \\ 4x_1 - 2x_2 + 3x_3 &= 12 \\ 0x_1 - 2x_2 + 2x_3 &= 6 \end{aligned}$$

- **ans:** Gauss elimination:

$$\begin{pmatrix} 2 & 1 & 1 & | & 3 \\ -2 & 1 & 0 & | & -3 \\ 4 & -2 & 3 & | & 12 \\ 0 & -2 & 2 & | & 6 \end{pmatrix}$$

$-2r_1 + r_3 \to r_3$, $r_1 + r_2 \to r_2$,

$$\begin{pmatrix} 2 & 1 & 1 & | & 3 \\ & 2 & 1 & | & 0 \\ & -4 & 1 & | & 6 \\ 0 & -2 & 2 & | & 6 \end{pmatrix}$$

$2r_2 + r_3 \to r_3$, $r_2 + r_4 \to r_4$,

$$\begin{pmatrix} 2 & 1 & 1 & | & 3 \\ & 2 & 1 & | & 0 \\ & & 3 & | & 6 \\ 0 & & 3 & | & 6 \end{pmatrix}$$

$-r_3 + r_4 \to r_4$,

$$\begin{pmatrix} 2 & 1 & 1 & | & 3 \\ & 2 & 1 & | & 0 \\ & & 3 & | & 6 \\ 0 & & 0 & | & 0 \end{pmatrix}$$

Backward substitution,

$$x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Gauss-Jordan elimination:

$$\begin{pmatrix} 2 & 1 & 1 & | & 3 \\ -2 & 1 & 0 & | & -3 \\ 4 & -2 & 3 & | & 12 \\ 0 & -2 & 2 & | & 6 \end{pmatrix}$$

$-2r_1 + r_3 \to r_3$, $r_1 + r_2 \to r_2$,

$$\begin{pmatrix} 2 & 1 & 1 & | & 3 \\ & 2 & 1 & | & 0 \\ & -4 & 1 & | & 6 \\ 0 & -2 & 2 & | & 6 \end{pmatrix}$$

$2r_2 + r_3 \to r_3$, $r_2 + r_4 \to r_4$,

$$\begin{pmatrix} 2 & 1 & 1 & | & 3 \\ & 2 & 1 & | & 0 \\ & & 3 & | & 6 \\ 0 & & 3 & | & 6 \end{pmatrix}$$

$-r_3 + r_4 \to r_4$,

$$\begin{pmatrix} 2 & 1 & 1 & | & 3 \\ & 2 & 1 & | & 0 \\ & & 3 & | & 6 \\ 0 & & 0 & | & 0 \end{pmatrix}$$

$\frac{1}{2}r_1 \to r_1$, $\frac{1}{2}r_2 \to r_2$, $\frac{1}{3}r_3 \to r_3$,

$$\begin{pmatrix} 1 & 1/2 & 1/2 & | & 3/2 \\ & 1 & 1/2 & | & 0 \\ & & 1 & | & 2 \\ 0 & & 0 & | & 0 \end{pmatrix}$$

$-\frac{1}{2}r_3 + r_1 \to r_1$, $\frac{1}{3}r_3 + r_2 \to r_2$,

$$\begin{pmatrix} 1 & 1/2 & & | & 1/2 \\ & 1 & & | & -1 \\ & & 1 & | & 2 \\ 0 & & 0 & | & 0 \end{pmatrix}$$

$-\frac{1}{2}r_2 + r_1 \to r_1$

$$\begin{pmatrix} 1 & & & | & 1 \\ & 1 & & | & -1 \\ & & 1 & | & 2 \\ 0 & & 0 & | & 0 \end{pmatrix}$$

It shows the solution,

$$x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

2. Solve the system $Ax = b$ by the Gaussian elimination.

$$A = \begin{pmatrix} 1 & 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 2 & 2 \\ 0 & -1 & 1 & -2 & 0 \end{pmatrix}, \; b = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

- **ans:** We do row operations on the matrix.

$$\begin{pmatrix} 1 & 1 & -1 & 2 & 1 & | & 1 \\ 2 & 1 & -1 & 2 & 2 & | & 3 \\ 0 & -1 & 1 & -2 & 0 & | & 1 \end{pmatrix}$$

$$\xrightarrow{-2r_1+r_2} \begin{pmatrix} 1 & 1 & -1 & 2 & 1 & | & 1 \\ & -1 & 1 & -2 & 0 & | & 1 \\ 0 & -1 & 1 & -2 & 0 & | & 1 \end{pmatrix}$$

$$\xrightarrow{-r_2+r_3} \begin{pmatrix} 1 & 1 & -1 & 2 & 1 & | & 1 \\ & -1 & 1 & -2 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Therefore, we can let x_3, x_4, x_5 be free variables. The solution would be (choosing 1 for one of x_3, x_4, x_5, and 0 for the other two, then solving for x_1 and x_2)

$$x = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + C_1 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

```
a=[1 1 -1 2 1]
b=[1 0 0 0 1]
A=[a; b+a; b-a]
x=[1 0 1 0 1]';
A*x
```

3. Find A^{-1}.

$$\begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & -1 \\ 3 & 10 & 6 \end{pmatrix}$$

- **ans:** We do row operations to transform the matrix

$$[A|I] \to [I|A^{-1}]$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & & \\ -2 & -3 & -1 & & 1 & \\ 3 & 10 & 6 & & & 1 \end{array}\right)$$

$$\xrightarrow{2r_1+r_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & & \\ & 1 & 1 & 2 & 1 & \\ 3 & 10 & 6 & & & 1 \end{array}\right)$$

$$\xrightarrow{-3r_1+r_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & & \\ & 1 & 1 & 2 & 1 & \\ & 4 & 3 & -3 & & 1 \end{array}\right)$$

$$\xrightarrow{-4r_2+r_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & & \\ & 1 & 1 & 2 & 1 & \\ & & -1 & -11 & -4 & 1 \end{array}\right)$$

$$\xrightarrow{(-1)r_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & & \\ & 1 & 1 & 2 & 1 & \\ & & 1 & 11 & 4 & -1 \end{array}\right)$$

$$\xrightarrow{(-1)r_3+r_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & & \\ & 1 & & -9 & -3 & 1 \\ & & 1 & 11 & 4 & -1 \end{array}\right)$$

$$\xrightarrow{(-1)r_3+r_1} \left(\begin{array}{ccc|ccc} 1 & 2 & & -10 & -4 & 1 \\ & 1 & & -9 & -3 & 1 \\ & & 1 & 11 & 4 & -1 \end{array}\right)$$

$$\xrightarrow{(-2)r_2+r_1} \left(\begin{array}{ccc|ccc} 1 & & & 8 & 2 & -1 \\ & 1 & & -9 & -3 & 1 \\ & & 1 & 11 & 4 & -1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 8 & 2 & -1 \\ -9 & -3 & 1 \\ 11 & 4 & -1 \end{pmatrix}$$

```
L=[1 0 0;-2 1 0; 3 4 1]
U=[1 2 1;0 1 1; 0 0 -1]
A=L*U, det(A)
inv(A)
```

4. (1) Use the Gauss elimination to solve the system.
(2) Find the LU decomposition of A, and use it to solve the system.
(3) Find the A^{-1}, and use it to solve the system.

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 3 \\ -2x_1 + x_2 + 0x_3 &= -3 \\ 4x_1 - 2x_2 + 3x_3 &= 12 \end{aligned}$$

- **ans:** (1) Gauss elimination:

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ -2 & 1 & 0 & -3 \\ 4 & -2 & 3 & 12 \end{array}\right)$$

$-2r_1 + r_3 \to r_3$, $r_1 + r_2 \to r_2$,

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ & 2 & 1 & 0 \\ & -4 & 1 & 6 \end{array}\right)$$

$2r_2 + r_3 \to r_3$,

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ & 2 & 1 & 0 \\ & & 3 & 6 \end{array}\right)$$

Backward substitution,

$$x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

(2) LU decomposition: We repeat GE steps, but record the negative row operations:

$$\begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & 0 \\ 4 & -2 & 3 \end{pmatrix}$$

$-2r_1 + r_3 \to r_3$, $r_1 + r_2 \to r_2$,

$$\begin{pmatrix} 2 & 1 & 1 \\ \boxed{-1} & 2 & 1 \\ \boxed{2} & -4 & 1 \end{pmatrix}$$

$2r_2 + r_3 \to r_3$,

$$\begin{pmatrix} 2 & 1 & 1 \\ \boxed{-1} & 2 & 1 \\ \boxed{2} & \boxed{-2} & 3 \end{pmatrix}$$

So we can read out

$$L = \begin{pmatrix} 1 & & \\ -1 & 1 & \\ 2 & -2 & 1 \end{pmatrix}, \; U = \begin{pmatrix} 2 & 1 & 1 \\ & 2 & 1 \\ & & 3 \end{pmatrix}$$

We can check the product of LU is really A itself.

Solving the system in two steps:

$$Ly = b, \; \begin{pmatrix} 1 & & \\ -1 & 1 & \\ 2 & -2 & 1 \end{pmatrix} y = \begin{pmatrix} 3 \\ -3 \\ 12 \end{pmatrix}$$

$$y = \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix}$$

$$Ux = y, \; \begin{pmatrix} 2 & 1 & 1 \\ & 2 & 1 \\ & & 3 \end{pmatrix} x = \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

(3) <u>Find A^{-1}:</u>

$$\left(\begin{array}{ccccccc} 2 & 1 & 1 & | & 1 & & \\ -2 & 1 & 0 & | & 0 & 1 & \\ 4 & -2 & 3 & | & & & 1 \end{array}\right)$$

$-2r_1 + r_3 \to r_3$, $r_1 + r_2 \to r_2$,

$$\left(\begin{array}{ccccccc} 2 & 1 & 1 & | & 1 & & \\ & 2 & 1 & | & 1 & 1 & \\ & -4 & 1 & | & -2 & & 1 \end{array}\right)$$

$2r_2 + r_3 \to r_3$,

$$\left(\begin{array}{ccccccc} 2 & 1 & 1 & | & 1 & & \\ & 2 & 1 & | & 1 & 1 & \\ & & 3 & | & 0 & 2 & 1 \end{array}\right)$$

$\frac{1}{2}r_1 \to r_1$, $\frac{1}{2}r_2 \to r_2$, $\frac{1}{3}r_3 \to r_3$,

$$\left(\begin{array}{ccccccc} 1 & 1/2 & 1/2 & | & 1/2 & & \\ & 1 & 1/2 & | & 1/2 & 1/2 & \\ & & 1 & | & 0 & 2/3 & 1/3 \end{array}\right)$$

$-\frac{1}{2}r_3 + r_1 \to r_1$, $-\frac{1}{2}r_3 + r_2 \to r_2$,

$$\left(\begin{array}{ccccccc} 1 & 1/2 & & | & 1/2 & -1/3 & -1/6 \\ & 1 & & | & 1/2 & 1/6 & -1/6 \\ & & 1 & | & 0 & 2/3 & 1/3 \end{array}\right)$$

$-\frac{1}{2}r_2 + r_1 \to r_1$,

$$\left(\begin{array}{ccccccc} 1 & & & | & 1/4 & -5/12 & -1/12 \\ & 1 & & | & 1/2 & 1/6 & -1/6 \\ & & 1 & | & 0 & 2/3 & 1/3 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1/4 & -5/12 & -1/12 \\ 1/2 & 1/6 & -1/6 \\ 0 & 2/3 & 1/3 \end{pmatrix}$$

Solution,

$$\begin{aligned} x = A^{-1}b &= \\ &= \begin{pmatrix} 1/4 & -5/12 & -1/12 \\ 1/2 & 1/6 & -1/6 \\ 0 & 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \\ 12 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \end{aligned}$$

5. Let

$$A = \begin{pmatrix} 1 & -3 & 2 \\ -1 & 0 & 2 \\ 1 & -1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}.$$

(a) Use Gaussian elimination to solve $Ax = b$.

(b) Find $A = LU$ where L is the unit lower triangular matrix. Using LU decomposition of A to solve $Ax = b$ again.

- <u>**ans:**</u>

(a)

$$(A|b) \to (U|c)$$

$r_1 + r_2 \to r_2$, $-r_1 + r_3 \to r_3$:

$$\left(\begin{array}{ccccc} 1 & -3 & 2 & | & 5 \\ 0 & -3 & 4 & | & 7 \\ 0 & 2 & -3 & | & -5 \end{array}\right)$$

$(2/3)r_2 + r_3 \to r_3$:

$$\left(\begin{array}{ccccc} 1 & -3 & 2 & | & 5 \\ 0 & -3 & 4 & | & 7 \\ 0 & 0 & -1/3 & | & -1/3 \end{array}\right)$$

$$x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

(b) Method 1 (easy to understand, but by the computer way we write less).

$$(I|A) \to (L^{-1}|U)$$

$$\left(\begin{array}{ccccccc} 1 & & & | & 1 & -3 & 2 \\ & 1 & & | & -1 & 0 & 2 \\ & & 1 & | & 1 & -1 & -1 \end{array}\right)$$

$r_1 + r_2 \to r_2$, $-r_1 + r_3 \to r_3$:

$$\left(\begin{array}{ccccccc} 1 & & & | & 1 & -3 & 2 \\ 1 & 1 & & | & 0 & -3 & 4 \\ -1 & & 1 & | & 0 & 2 & -3 \end{array}\right)$$

$(2/3)r_2 + r_3 \to r_3$:

$$\begin{pmatrix} 1 & & & | & 1 & -3 & 2 \\ 1 & 1 & & | & 0 & -3 & 4 \\ -1 & 2/3 & 1 & | & 0 & 0 & -1/3 \end{pmatrix}$$

To find L from L^{-1}, simply change the signs of all strictly lower triangular part of the matrix:

$$L = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ -1 & 2/3 & 1 \end{pmatrix}$$
$$L^{-1} = \begin{pmatrix} 1 & & \\ -1 & 1 & \\ 1 & -2/3 & 1 \end{pmatrix}$$

To check:

$$\begin{aligned} LU &= \begin{pmatrix} 1 & & \\ -1 & 1 & \\ 1 & -2/3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & -1/3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -3 & 2 \\ -1 & 0 & 2 \\ 1 & -1 & -1 \end{pmatrix} = A \end{aligned}$$

Method 2 (as if a computer does it, saving storage!). Use this method.

$r_1 + r_2 \to r_2$, $-r_1 + r_3 \to r_3$:

$$\begin{pmatrix} 1 & -3 & 2 \\ \boxed{-1} & -3 & 4 \\ \boxed{1} & 2 & -3 \end{pmatrix}$$

$(2/3)r_2 + r_3 \to r_3$:

$$\begin{pmatrix} 1 & -3 & 2 \\ \boxed{-1} & -3 & 4 \\ \boxed{1} & \boxed{-2/3} & -1/3 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & & \\ -1 & 1 & \\ 1 & -2/3 & 1 \end{pmatrix}$$

U is the upper triangular matrix above.

Two steps to solve $Ax = LUx = b$. First forward substitution:

$$Ly = b, y = \begin{pmatrix} 5 \\ 7 \\ -1/3 \end{pmatrix}$$

Then backward substitution:

$$Ux = y, x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

6. Let A and B be 3×3 matrices with $\det A = -3$ and $\det B = 2$. Find

$$\det(A^T B),\ \det(-A),\ \det(2AB),\ \det(A^{-1}B)$$

- **ans:** Please note three important formulas.

$$\begin{aligned} \det(AB) &= \det(A)\det(B), \\ \det(A^T) &= \det(A), \\ \det(A^{-1}) &= \frac{1}{\det(A)} \end{aligned}$$

Another point is that we can factor out a constant out of a row, or of a column, from the determinant. So $\det(cA) = c^n \det(A)$ if A is $n \times n$.

$$\begin{aligned} \det(A^T B) &= \det(A^T)\det(B) = \det(A)\det(B) = -6 \\ \det(-A) &= (-1)^3 \det(A) = 3 \\ \det(2AB) &= 2^3 \det(AB) = 8\det(A)\det(B) = -48 \\ \det(A^{-1}B) &= \det(A)^{-1}\det(B) = -\frac{2}{3} \end{aligned}$$

7. Find the determinant ($\det A$), its singularity and its inverse (A^{-1}) if it exists.

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & 2 \end{pmatrix}$$

- **ans:** Find A^{-1}

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$-r_1 + r_2$, $-r_1 + r_3$,

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ & 0 & 0 & -1 \\ & 1 & 0 & 2 \\ 0 & 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ -1 & & 1 & \\ & & & 1 \end{pmatrix}$$

switch r_2 and r_3,

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ & 1 & 0 & 2 \\ & 0 & 0 & -1 \\ 0 & 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & & & \\ -1 & & 1 & \\ -1 & 1 & & \\ & & & 1 \end{pmatrix}$$

$-r_2 + r_4$,

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ & 1 & 0 & 2 \\ & 0 & 0 & -1 \\ 0 & & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & & \\ -1 & & 1 & \\ -1 & 1 & & \\ 1 & & -1 & 1 \end{pmatrix}$$

switch r_3 and r_4,

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ & 1 & 0 & 2 \\ & & -1 & 0 \\ & & & -1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ -1 & & 1 & \\ 1 & & -1 & 1 \\ -1 & 1 & & \end{pmatrix}$$

$-r_3$ and $-r_4$,

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ & 1 & 0 & 2 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ -1 & & 1 & \\ -1 & & 1 & -1 \\ 1 & -1 & & \end{pmatrix}$$

$-2r_3 + r_1$,

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ & 1 & 0 & 2 \\ 0 & & 1 & 0 \\ & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & & -2 & 2 \\ -1 & & 1 & \\ -1 & & 1 & -1 \\ 1 & -1 & & \end{pmatrix}$$

$-2r_4 + r_2$, $-r_4 + r_1$,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -2 & 2 \\ -3 & 2 & 1 & \\ -1 & & 1 & -1 \\ 1 & -1 & & \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 2 & 1 & -2 & 2 \\ -3 & 2 & 1 & 0 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

For the determinant, we can use direct expansions. Or we can use the matrix after first row operations above: $-r_1+r_2$, $-r_1 + r_3$,

$$\begin{aligned} \det A &= \det \begin{pmatrix} 1 & 0 & 2 & 1 \\ & 0 & 0 & -1 \\ & 1 & 0 & 2 \\ 0 & 1 & -1 & 2 \end{pmatrix} \\ &= \det \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & -1 & 2 \end{pmatrix} \\ &= (-1) \det \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \\ &= (-1)(1) \det \begin{pmatrix} -1 \end{pmatrix} = 1 \end{aligned}$$

```
A=[1 0 2 1;  1 0  2 0; 1 1 2 3;0 1 -1 2]
det(A),
inv(A)
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8. Find $\det A$, A^{-1}, and use A^{-1} to solve $Ax = b$.

$$A = \begin{pmatrix} 1 & -3 & 2 \\ -1 & 0 & 2 \\ 1 & -1 & -1 \end{pmatrix}, \; b = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

- **<u>ans:</u>** Add 2 times column 1 to column 3, then expand it by row 2.

$$\det A = \begin{vmatrix} 1 & -3 & 4 \\ -1 & 0 & 0 \\ 1 & -1 & 1 \end{vmatrix} = (-1)(-1) \begin{vmatrix} -34 & \\ -1 & -1 \end{vmatrix} = 4 - 3 = 1$$

$$\begin{pmatrix} 1 & -3 & 2 & | & 1 & & \\ -1 & 0 & 2 & | & & 1 & \\ 1 & -1 & -1 & | & & & 1 \end{pmatrix}$$

$$\overset{r_1+r_2}{\longrightarrow} \begin{pmatrix} 1 & -3 & 2 & | & 1 & & \\ & -3 & 4 & | & 1 & 1 & \\ 1 & -1 & -1 & | & & & 1 \end{pmatrix}$$

$$\overset{-r_1+r_3}{\longrightarrow} \begin{pmatrix} 1 & -3 & 2 & | & 1 & & \\ & -3 & 4 & | & 1 & 1 & \\ & 2 & -3 & | & -1 & & 1 \end{pmatrix}$$

$$\overset{(-1/3)r_2}{\longrightarrow} \begin{pmatrix} 1 & -3 & 2 & | & 1 & & \\ & 1 & -4/3 & | & -1/3 & -1/3 & \\ & 2 & -3 & | & -1 & & 1 \end{pmatrix}$$

$$\overset{3r_2+r_1}{\longrightarrow} \begin{pmatrix} 1 & & -2 & | & 0 & -1 & \\ & 1 & -4/3 & | & -1/3 & -1/3 & \\ & 2 & -3 & | & -1 & & 1 \end{pmatrix}$$

$$\overset{-2r_2+r_3}{\longrightarrow} \begin{pmatrix} 1 & & -2 & | & 0 & -1 & \\ & 1 & -4/3 & | & -1/3 & -1/3 & \\ & & -1/3 & | & -1/3 & 2/3 & 1 \end{pmatrix}$$

$$\overset{(-3)r_3}{\longrightarrow} \begin{pmatrix} 1 & & -2 & | & 0 & -1 & \\ & 1 & -4/3 & | & -1/3 & -1/3 & \\ & & 1 & | & 1 & -2 & -3 \end{pmatrix}$$

$$\overset{(4/3)r_3+r_2}{\longrightarrow} \begin{pmatrix} 1 & & -2 & | & 0 & -1 & \\ & 1 & & | & 1 & -3 & -4 \\ & & 1 & | & 1 & -2 & -3 \end{pmatrix}$$

$$\overset{2r_3+r_1}{\longrightarrow} \begin{pmatrix} 1 & & -2 & | & 2 & -5 & -6 \\ & 1 & & | & 1 & -3 & -4 \\ & & 1 & | & 1 & -2 & -3 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 2 & -5 & -6 \\ 1 & -3 & -4 \\ 1 & -2 & -3 \end{pmatrix}$$

$$x = A^{-1}b = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Since $\det A = 1$, we know the entries of A^{-1} would be all integers. We can avoid fractions above in finding A^{-1}. Let us try it again.

$$
\begin{pmatrix}
1 & -3 & 2 & | & 1 & & \\
-1 & 0 & 2 & | & & 1 & \\
1 & -1 & -1 & | & & & 1
\end{pmatrix}
$$

$$
\overset{r_3+r_2}{\longrightarrow}
\begin{pmatrix}
1 & -3 & 2 & | & 1 & & \\
 & -1 & 1 & | & & 1 & 1 \\
1 & -1 & -1 & | & & & 1
\end{pmatrix}
$$

$$
\overset{-r_3+r_1}{\longrightarrow}
\begin{pmatrix}
 & -2 & 3 & | & 1 & & -1 \\
 & -1 & 1 & | & & 1 & 1 \\
1 & -1 & -1 & | & & & 1
\end{pmatrix}
$$

$$
\overset{(-2)r_2+r_1}{\longrightarrow}
\begin{pmatrix}
 & & 1 & | & 1 & -2 & -3 \\
 & -1 & 1 & | & & 1 & 1 \\
1 & -1 & -1 & | & & & 1
\end{pmatrix}
$$

$$
\overset{-r_1+r_2\to r_2, r_1+r_3\to r_3}{\longrightarrow}
\begin{pmatrix}
 & & 1 & | & 1 & -2 & -3 \\
 & -1 & & | & -1 & 3 & 4 \\
1 & -1 & & | & 1 & -2 & -2
\end{pmatrix}
$$

$$
\overset{-r_2+r_3\to r_3}{\longrightarrow}
\begin{pmatrix}
 & & 1 & | & 1 & -2 & -3 \\
 & -1 & & | & -1 & 3 & 4 \\
1 & & & | & 2 & -5 & -6
\end{pmatrix}
$$

$$
\overset{(-1)r_2\to r_2, r_1\leftrightarrow r_3}{\longrightarrow}
\begin{pmatrix}
1 & & & | & 2 & -5 & -6 \\
 & 1 & & | & 1 & -3 & -4 \\
 & & 1 & | & 1 & -2 & -3
\end{pmatrix}
$$

$$
A^{-1} = \begin{pmatrix}
2 & -5 & -6 \\
1 & -3 & -4 \\
1 & -2 & -3
\end{pmatrix}
$$

9. Determine whether the following are subspaces of P_4. (P_4 is the set of all polynomials of degree 4 or less)
   (a) The set of polynomials in P_4 of even degree
   (b) The set of polynomials of degree 3
   (c) The set of all polynomials $p(x)$ in P_4 such that $p(0) = 0$.
   (d) The set of all polynomials in P_4 having at least one real root.

   - **ans:**

   (a) No. $x + x^2$ and $x - x^2$ are both in the set. But the sum $2x$ is not of even degree, and it is not in the set.

   (b) No. x^3 and $-x^3$ are both of degree 3. But the sum is not.

   (c) Yes. If p_1 and p_2 in the set, then $p_1 + p_2$ and Cp_1 are both in the set as they are polynomials of degree at most 4 and have 0 value at $x = 0$.

   (d) No. $p_1(x) = x + 1$ and $p_2(x) = x^2 - x$ are both in the set. But the sum $x^2 + 1$ has no real root, and it is not in the set.

10. Determine whether the following sets form subspaces of R^3.
    (a) $\{(x_1, x_2, x_3)^T \mid x_1 + 2x_2 - x_3 = 0\}$
    (b) $\{(x_1, x_2, x_3)^T \mid x_1 = x_2 = 1\}$
    (c) $\{(x_1, x_2, x_3)^T \mid x_1 = 2x_3\}$
    (d) $\{(x_1, x_2, x_3)^T \mid x_3 = x_2 + 1\}$
    (e) $\{(x_1, x_2, x_3)^T \mid x_3 = 0, \text{ or } x_1 = -x_2\}$

    - **ans:**

    (a) Yes. It is the solution space (nullspace $Ax = 0$, $A = \begin{pmatrix}1 & 2 & -1\end{pmatrix}$).

    $$\{c_1 \begin{pmatrix}1\\0\\1\end{pmatrix} + c_2 \begin{pmatrix}2\\-1\\0\end{pmatrix}\}$$

    We can also check $C1$ and $C2$ directly.

    (b) No. $(1, 1, 0)^T$ belongs to the set, but $2(1, 1, 0)^T = (2, 2, 0)^T$ does not. C1 failed.

    (c) Yes. It is the solution space (nullspace $Ax = 0$, $A = \begin{pmatrix}1 & 0 & -2\end{pmatrix}$).

    $$\{c_1 \begin{pmatrix}2\\0\\1\end{pmatrix} + c_2 \begin{pmatrix}0\\1\\0\end{pmatrix}\}$$

    We can also check $C1$ and $C2$ directly.

    (d) No. $(0, 0, 1)^T$ belongs to the set, but $2(0, 0, 1)^T = (0, 0, 2)^T$ does not. C1 failed.

    (e) No. $(1, 1, 0)^T$ and $(1, -1, 1)^T$ belong to the set, but $(1, 1, 0)^T + (1, -1, 1)^T = (2, 0, 1)^T$ does not. C2 failed.

11. Given

    $$x_1 = \begin{pmatrix}2\\1\\3\end{pmatrix}, \; x_2 = \begin{pmatrix}3\\-1\\4\end{pmatrix}, \; x_3 = \begin{pmatrix}2\\6\\4\end{pmatrix}$$

    (a) Show that x_1, x_2, x_3 are linearly dependent.
    (b) Show that x_1, x_2 are linearly dependent.
    (c) What is the dimension of $\text{Span}(x_1, x_2, x_3)$?
    (d) Give a geometric description of $\text{Span}(x_1, x_2, x_3)$.

    - **ans:**

    (a) As we have precisely 3 vectors, we can simply check the determinant.

    $$\begin{vmatrix}2 & 3 & 2\\1 & -1 & 6\\3 & 4 & 4\end{vmatrix}$$

    $$=2(-1)4 - 2(6)4 + 3(6)3 - 3(1)(4) + 2(1)(4) - 2(-1)(3)$$
    $$= -8 - 48 + 54 - 12 + 8 + 6 = 68 - 68 = 0$$

    So the three vectors are linearly dependent.

    (b) We solve the homogeneous system, $Ax = 0$.

    $$\begin{pmatrix}2 & 3 & | & 0\\1 & -1 & | & 0\\3 & 4 & | & 0\end{pmatrix}$$

    $$\to \begin{pmatrix}2 & 3 & | & 0\\ & -5/2 & | & 0\\ & -1/2 & | & 0\end{pmatrix}$$

So we have only zero solution $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. So the two vectors are linearly dependent.

(c) By (a), the dimension of $\mathrm{Span}(x_1, x_2, x_3)$ is less than 3. By (b), as we found two linearly independent vectors in $\mathrm{Span}(x_1, x_2, x_3)$, the dimension of $\mathrm{Span}(x_1, x_2, x_3)$ is at least 2. So the dimension of $\mathrm{Span}(x_1, x_2, x_3)$ is 2.

(d) By above work, we know the third vector is a linear combination of the first two vectors. The $\mathrm{Span}(x_1, x_2, x_3)$ $=\mathrm{Span}(x_1, x_2)$. All vectors in $\mathrm{Span}(x_1, x_2, x_3)$ are in the format

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + b \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

for any real numbers a and b.

We can eliminate a and b by substituting the first and the second equation into the third equation. I.e., by adding -2 times the second equation to the first equation,

$$x - 2y = 5b$$

by adding 2 times the second equation to the first equation,

$$x + 3y = 5a$$

Put them into the third equation,

$$z = \frac{x + 3y}{5}(3) + \frac{x - 2y}{5}(4)$$

This is an equation for a plane in 3D, (must passing the origin).

$$7x + y - 5z = 0$$

(we can verify that the given three vectors all satisfy the plane equation.)

12. Find the transition matrix from $[\mathbf{v}_1, \mathbf{v}_2]$ to $[\mathbf{u}_1, \mathbf{u}_2]$.

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

Find the coordinates of $\mathbf{v} = 1\mathbf{v}_1 + 3\mathbf{v}_2$ under the basis $[\mathbf{u}_1, \mathbf{u}_2]$

- **ans:** The transition matrix

$$\begin{aligned} S &= U^{-1}V \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{v}_u &= S\mathbf{v}_v \\ &= \begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 11 \\ -4 \end{pmatrix} \end{aligned}$$

To check:

$$U\mathbf{v}_u = V\mathbf{v}_v$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

```
u1=[1 1]'; u2=[2 1]';U=[u1 u2]
u1=[0 1]'; u2=[1 2]';V=[u1 u2]
iU=inv(U)
uv=iU*V, v=[1 3]'
uv*v
U*ans, V*v
```

13. Determine if $\mathbf{b}$ is in the column space of A and state whether the system $A\mathbf{x} = \mathbf{b}$ is consistent. If the system is consistent, determine whether there will be one or infinitely many solutions.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

- **ans:** 3.6: 45e We need to solve the linear system $[A|b]$.

$$\begin{pmatrix} 0 & 1 & | & 2 \\ 1 & 0 & | & 5 \\ 0 & 1 & | & 2 \end{pmatrix}$$

It is easy to see there is a solution

$$x = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

So the system is consistent and $\mathbf{b}$ is in the column space of A. The solution is unique.

```
a=[ ]; b=[ ]';
A=[a b]
A(2,:)=A(2,:)-1*A(1,:)
A(3,:)=A(3,:)-1*A(1,:)
```

14. Let A be the 3×5 matrix whose columns are

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{a}_2 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{a}_5 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

(a) Find the dimension of $\mathrm{Span}\{\mathbf{a}_1, ..., \mathbf{a}_5\}$, and a basis for the span.

(b) Find the rank of A. Find its dimension and a basis for $N(A)$.

(c) Find a basis for the row space of A.

(d) Extend $\{\mathbf{a}_1, \mathbf{a}_5\}$ to a basis for R^3 .

- **ans:**

(a) We reduce A to upper triangular matrix, and identify the rows and columns of pivots, which will tell the independent rows and columns of the original A. Also from the reduced upper triangular matrix, we can easily solve $Ax = 0$ to find $N(A)$.

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 & 2 \\ 1 & -1 & 0 & 1 & 1 \end{pmatrix} \overset{-r_1+r_2, -r_1+r_3}{\rightarrow} \begin{pmatrix} 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So the first vector and the third vector are linearly independent. Which forms a basis for Span$\{\mathbf{a}_1, ..., \mathbf{a}_5\}$:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\dim \mathrm{Span}\{\mathbf{a}_1, ..., \mathbf{a}_5\} = \dim \mathrm{Col}(A) = 2$$

(b) The rank is the same as $\dim \mathrm{Col}(A)$, which is 2.

Now it is easy to do backward substitution to find the solution of homogeneous system $Ax = 0$, by the above row echelon form. These solutions x form the nullspace. Here, other than the pivot variables, x_1 and x_3, we can set the other three variables, as free variables.

$$\begin{aligned} x = & \begin{pmatrix} c_1 - c_2 - c_3 \\ c_1 \\ c_2 - c_3 \\ c_2 \\ c_3 \end{pmatrix} \\ = & c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

So

$$N(A) = \mathrm{Span}\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \}$$

We note that the rank of A plus the dim of Col(A) is always the number of columns in A.

(c) Note that we did not do row permutations here. This way, we can link the original independent rows to the final independent rows. From the row echelon form of A above, we know the first two rows of A are linearly independent which form a basis for

$$\mathrm{Row}(A) = \mathrm{Span}\{ \begin{pmatrix} 1 & -1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 & 0 & 2 \end{pmatrix} \}$$

(d) We can extend the set to a basis by inspection, adding another linear indepdent vector. We can also do it "officially" by adding the standard basis plus paring 5 vectors down to 3 to get a basis.

Method 1, by inspection, we add vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. We verify the basis (linear independence) by computing the determinant (we did one column operation):

$$\begin{aligned} \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_3 & \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1 \neq 0 \end{aligned}$$

Method 2. paring down $\{\mathbf{a}_1, \mathbf{a}_5, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to a basis for R^3.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$-r_1 + r_2$, $-r_1 + r_3$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ & 1 & -1 & 1 & 0 \\ & & -1 & 0 & 1 \end{pmatrix}$$

The first three columns are linear independent vectors. We pick up the original columns 1, 2, and 3, to get an pared down basis:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$