**M341 Study Guide for E 2 (quiz 4, due Wed)** (S. Zhang) .

1. Show the linearly dependence (1) by Wronskian, (2) directly by finding a zero linear combination with nonzero coefficients.

$$f(x) = 2x - x^2,\ g(x) = 3x^2,\ h(x) = 2x + 2x^2$$

- **ans:** Method 1, (Wronskian)

$$W(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = \begin{vmatrix} 2x - x^2 & 3x^2 & 2x + 2x^2 \\ 2 - 2x & 6x & 2 + 4x \\ -2 & 6 & 4 \end{vmatrix} = 0$$

Method 2, (coefficients)

$$Af + Bg + Ch = 0$$

$$A2 + 0 + 2C = 0,\ -A + 3B + 2C = 0$$

Let $C = -1$,

$$1(2x - x^2) + 1(3x^2) + (-1)(2x + 2x^2) = 0$$

2. Show the linearly dependence

$$f(x) = |x| - x,\ g(x) = |x| + x,\ h(x) = x$$

- **ans:** We cannot use Wronskian here.

$$Af + Bg + Ch = 0$$

$$(A + B)|x| + (-A + B + C)x = 0$$

Let $A = 1$, then $B = -1$, and $C = 2$.

$$1(|x| - x) + (-1)(|x| + x) + 2(x) = 0$$

3. Given one solution $y_1 = e^x$, reduce the equation to a first order equation and then find the general solution of the second equation:

$$(x + 1)y'' - (x + 2)y' + y = 0.$$

- **ans:** $y_2 = vy_1$. Plug $y = ve^x$, into the equation:

$$(x + 1)(v'' + 2v') - (x + 2)v' = 0$$

$$(x + 1)\frac{dv'}{dx} = -xv'$$

$$\frac{dv'}{v'} = \frac{-x}{x+1}dx = (-1 + \frac{1}{x+1})dx$$

$$\int \frac{dv'}{v'} = \int (-1 + \frac{1}{x+1})dx$$

$$\ln v' = -x + \ln(x + 1)$$

$$v' = e^{-x+\ln(x+1)} = e^{-x}(x + 1)$$

$$v = \int e^{-x}(x + 1)dx$$

We use the integration by parts method, letting $u = (x + 1)$, $dw = e^{-x}$, $du = dx$, $w = -e^{-x}$

$$\int udw = uw - \int wdu$$

$$v = -e^{-x}(x + 1) - \int -e^{-x}dx = -e^{-x}(x + 1) - e^{-x} = e^{-x}(x + 2)$$

$$y_2 = vy_1 = ve^x = (x + 2)$$

$$y = C_1y_1 + C_2y_2 = C_1e^x + C_2(2 + x)$$

4. Given one solution, find the general solution.

$$9y''' + 11y'' + 4y' - 14y = 0 = 0,\ y = e^{-x}\sin x$$

- **ans:** We need to find the roots of

$$9r^3 + 11r^2 + 4r - 14 = 0$$

From the given solution $y = e^{-x}\sin x$ we get two roots.

$$r = -1 \pm i\ \Rightarrow (r + 1)^2 + 1 = 0$$

By doing a division, we get the third root:

$$9r^3 + 11r^2 + 4r - 14 = ((r + 1)^2 + 1)(9r - 7)$$

$$y = c_1e^{-x}\cos x + c_2e^{-x}\sin x + c_3e^{7x/9}$$

5. Find a linear homogeneous constant-coefficient equation so that the function is a solution.

$$y(x) = e^{2x}x\sin 2x + 4\cos 2x - x^2\sin 2x$$

- **ans:** Roots are

$$r = 2 \pm 2i, 2 \pm 2i, \pm 2i, \pm 2i, \pm 2i$$

$$((r - 2)^2 + 4)^2(r^2 + 4)^3 = 0$$

DE:

$$((D - 2)^2 + 4)^2(D^2 + 4)^3y = 0$$

6. Find the general solution of the homogeneous equation $y_c$ and write down the form for undetermined coefficients for $y_p$. (Do not solve for $y_p$.)

$$y'' - 2y' + 10y = e^x\sin 3x - e^x + \cos 3x.$$

- **ans:** $r = 1 \pm 3i$,

$$y_c = e^xC_1\cos 3x + e^xC_2\sin 3x.$$

$$y_p = e^x(A\cos 3x + B\sin 3x)x + Ce^x + (D\cos 3x + F\sin 3x)$$

7. Find the general solution of the homogeneous equation $y_c$ and write down the form for undetermined coefficients for $y_p$. (Do not solve for $y_p$.)

$$\begin{aligned}&(D^2-4)^2(D-2)^2(D^2-2D+5)^2y\\ =&e^x\sin 3x - e^{2x}x+\cos 2x + e^x x^2\cos 2x.\end{aligned}$$

- **ans:** First, we find $y_H = y_c$. From the characteristic equation (replace $D$ by $r$):

$$(D^2-4)^2(D-2)^2(D^2-2D+5)^2=0$$

we get roots:

$$r=2,2,2,2,-2,-2,1\pm 2i,1\pm 2i$$

$$\begin{aligned}y_H=y_c=&e^{2x}(A+Bx+Cx^2+Dx^3)+e^{-2x}(E+Fx)\\ &+e^x((G+Hx)\cos 2x+(I+Jx)\sin 2x).\end{aligned}$$

Then we can write down a $y_p$ form, (note $1\pm 2i$ is a double root, and 2 is a 4-fold root)

$$\begin{aligned}y_p=&e^x(A\cos 3x+B\sin 3x)\\ &+(C+Dx)x^4e^{2x}+(E\cos 2x+F\sin 2x)\\ &e^x((G+Hx+Ix^2)\cos 2x+(J+Kx+Lx^2)\sin 2x)x^2\end{aligned}$$

8. Suppose that the mass in a mass-spring-dash-pot system with $m = 25$, $c = 10$, and $k = 226$ is set in motion with $x(0) = 20$ and $x'(0) = 41$.
(a) Find the position function $x(t)$ and sketch its graph.
(b) Find the pseudo-period of the oscillations and the equations for the "envelop curves", i.e. time-varying amplitude.

- **ans:** EP §3.4: 14,

$$\begin{aligned}mx''&=-10x'-kx\\ 25r^2+10r+226&=(5r+1)^2+15^2\\ r&=-\frac{1}{5}\pm 3i\\ x(t)&=e^{-t/5}(20\cos 3t+15\sin 3t)\\ &=e^{-t/5}25\cos(3t-\alpha)\end{aligned}$$

where $\alpha=\tan^{-1}(3/4)=0.6435$

The time-varying amplitude is

$$x=25e^{-t/5}$$

The pseudoperiod of the oscillation is

$$T=\frac{2\pi}{3}$$

9. For the forced oscillation

$$mx''+cx'+kx=F(t)=F_0\cos\omega t$$

suppose

$$m=1,\ c=2,\ k=26,\ F_0=82,\ \omega=4$$

$$x(0)=6,\ x'(0)=0$$

Find the transient motion (i.e. $x_c$ part) and the steady periodic oscillation (i.e. $x_P = x_{sp}$).

Finally, find the practical resonance frequency $\omega$ for the given $m, c, k$, and $F_0$.

Given the time-varing amplitude:

$$\begin{aligned}C(\omega)&=\sqrt{A^2+B^2}\\ &=\frac{F_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}\end{aligned}$$

- **ans:**

$$x''+2x'+26x=82\cos 4t$$

For $x_c = x_H$,

$$r^2+2r+26=0;\ r=-1\pm 5i$$

$$x_c=e^{-t}(c_1\cos 5t+c_2\sin 5t)$$

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m:= 1; k:= 26; c:= 2; F0:= 82; w:= 4;
x0:=6: xp0:=0:
f:= rhs(dsolve({m*diff(x(t),t$2)+
 c*(diff(x(t),t))+k*x(t) = F0*cos(w*t),
 x(0)=x0, D(x)(0)=xp0}, x(t)));
plot(f,t=0..30);
```

Since $r = 4i$ is not a root, we set up an $x_P$ form

$$x_P=A\cos 4t+B\sin 4t$$

$$\begin{aligned}10A+8B&=82\\ -8A+10B&=0\end{aligned}$$

$$x_P=5\cos 4t+4\sin 4t$$

$$\begin{aligned}x=&x_H+x_P\\ =&e^{-t}(c_1\cos 5t+c_2\sin 5t)+5\cos 4t+4\sin 4t\end{aligned}$$

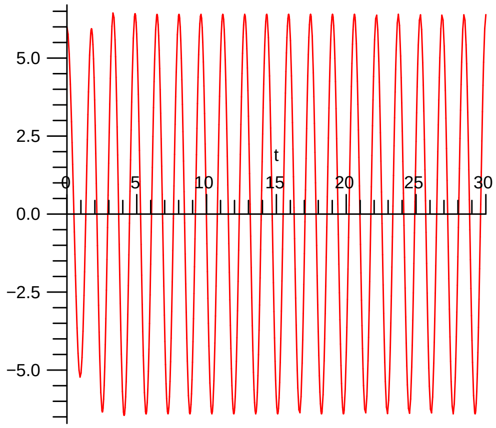
By the initial conditions,

$$x=e^{-t}(\cos 5t-3\sin 5t)+5\cos 4t+4\sin 4t$$

The transient motion (disappear after a while)

$$x_{tr}=e^{-t}(\cos 5t-3\sin 5t)$$

Only can be seen on the first cycle.




The steady periodic oscillation

$$\begin{aligned} x_{sp} =& 5\cos 4t + 4\sin 4t \\ =& \sqrt{41}\cos(4t-\delta) \\ \delta =& \tan^{-1}\frac{4}{5} \simeq 0.6747 \end{aligned}$$

<u>we can skip the theory and jump to formula</u>

Since $r = \omega i$ is not a root of the characteristic equation (because of friction, i.e., $c \neq 0$), we set

$$x_P = x_{sp} = A\cos\omega t + B\sin\omega t$$

Plut it into the equation

$$\begin{aligned} (k-m\omega^2)A + c\omega B &= F_0 \\ -c\omega A + (k-m\omega^2)B &= 0 \end{aligned}$$

We get

$$\begin{aligned} A &= \frac{(k-m\omega^2)F_0}{(k-m\omega^2)^2+(c\omega)^2} \\ B &= \frac{c\omega F_0}{(k-m\omega^2)^2+(c\omega)^2} \end{aligned}$$

$$\begin{aligned} x_P = x_{sp} &= A\cos\omega t + B\sin\omega t \\ &= C\cos(\omega t - \delta) \\ C(\omega) &= \sqrt{A^2+B^2} \\ &= \frac{F_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}} \end{aligned}$$

When $c$ is small and $k \sim m\omega^2$ (i.e., $\omega \sim \omega_0$), the amplitude is much bigger.

Plugging in the given data, we have

$$\begin{aligned} C(\omega) &= \frac{F_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}} \\ &= \frac{82}{\sqrt{676-48\omega^2+\omega^4}} \\ C'(\omega) &= \frac{-164\omega(\omega^2-24)}{(676-48\omega^2+\omega^4)^{3/2}} = 0 \\ \omega &= \sqrt{24} \end{aligned}$$

So when $\omega = \sqrt{24}$, we have a practical resonance.

10. In an RLC circuit with input voltage $E(t)$, find $I_sp(t)$ in the standard form.

$$R = 20\Omega,\ L = 5H,\ C = 0.01F;\ E(t) = 200\cos 5tV.$$

- **<u>ans:</u>**

$$LI'' + RI' + C^{-1}I = E'(t)$$

$$5I'' + 20I' + 100I = -1000\sin 5t$$

$$I'' + 4I' + 20I = -200\sin 5t$$

$$I_P = I_{sp} = A\cos 5t + B\sin 5t$$

$$-25A + 20B + 20A = 0,\ -25B - 20A + 20B = -200$$

$$-A + 4B = 0,\ -4A - B = -40$$

$$A = \frac{160}{7},\ \ B = \frac{40}{7}$$

$$\begin{aligned} I_P = I_{sp} &= \frac{160}{7}\cos 5t + \frac{40}{7}\sin 5t \\ &= \frac{40}{\sqrt{17}}\left(\frac{4}{\sqrt{17}}\cos 5t + \frac{1}{\sqrt{17}}\sin 5t\right) \\ &= \frac{40}{\sqrt{17}}\sin(5t-\delta) \\ \delta &= 2\pi - \tan^{-1}4 \simeq 4.95 \end{aligned}$$

11. Transform the differential equation into an equivalent system of first-order differential equations.

$$x'' - 2x' + 3x = 2t$$

- **<u>ans:</u>**

$$\begin{aligned} \mathbf{v}' &= \mathbf{f}(t, \mathbf{v}) \\ \mathbf{v} &= \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y' \end{pmatrix} \end{aligned}$$

$$\begin{aligned} x' &= y \\ y' &= -3x + 2y + 2t \end{aligned}$$

12. Solve the first order system by elimination

$$\begin{aligned} x' &= -y \qquad\qquad x(0) = 0 \\ y' &= 13x + 4y; y(0) = 3 \end{aligned}$$

- **ans:**

$$\begin{aligned} x'' = -y' &= -13x - 4y = -13x + 4x' \\ x'' - 4x' &+ 13x = 0 \end{aligned}$$

$$\begin{aligned} r &= 2 \pm 3i \\ x &= e^{2t}(c_1 \cos 3t + c_2 \sin 3t) \\ x(0) &= 0,\ c_1 = 0 \\ x &= e^{2t} c_2 \sin 3t \\ y &= x' = -e^{2t}(3c_2 \cos 3t + 2c_2 \sin 3t) \\ y(0) &= 3,\ c_2 = -1 \\ y &= x' = -e^{2t}(-3 \cos 3t - 2 \sin 3t) \\ x &= -e^{2t} \sin 3t \end{aligned}$$