M341 Study Guide for E 1 (S. Zhang) .

1. Find a differential equation $y' = f(x, y)$ so that a solution $y = g(x)$ has the described geometric property for its graph.

   (a) The slope of the graph of g at point (x, y) is the sum of x and y.

   (b) The line tangent to the graph of g at (x, y) passes through the point $(-y, x)$.

   - **ans:** Draw a graph each case.

   (a) The slope is y'.

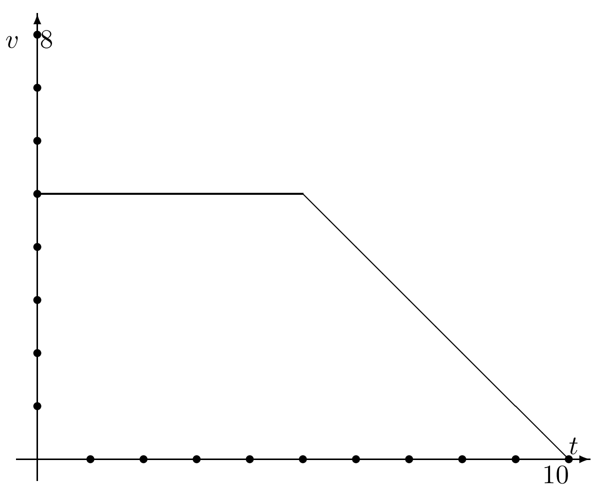
$$y' = x + y$$

   (b) The tangent line goes through two points (x, y) and $(-y, x)$. Therefore, its slope is (the ratio of y increment over x increment)

$$m = \frac{x - y}{-y - x}$$

$$y' = \frac{x - y}{-y - x}$$

2. Sketch the position function $x = x(t)$, assuming $x(0) = 0$ and given the velocity:

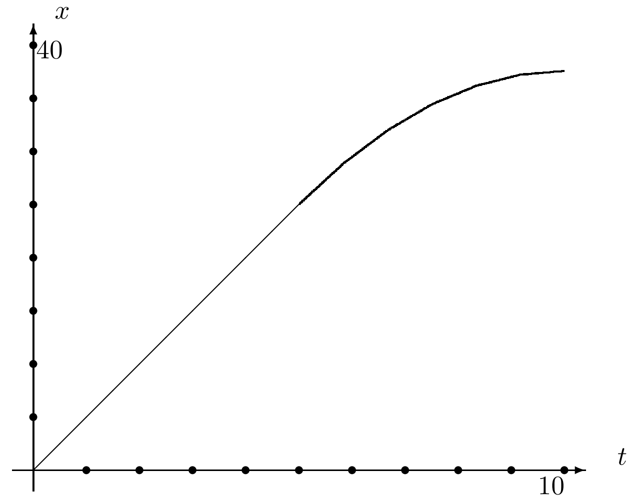


   - **ans:** Read from the graph.

$$v(t) = 5, \quad t \in [0, 5]$$
$$v(t) = 10 - t, \quad t \in [5, 10]$$

   Solution.

$$x(t) = \int 5 = 5t, \quad t \in [0, 5]$$
$$x(5) = 25$$
$$x(t) = \int 10 - t = 10t - \frac{1}{2}t^2 + C, \quad t \in [5, 10]$$
$$x(t) = 10t - \frac{1}{2}t^2 - \frac{25}{2}, \quad t \in [5, 10]$$



3. Determine if the existence and uniqueness theorem for the first order DE guarantee the solution (in a neighborhood) for each of the initial value problems:

$$(a)\ \frac{dy}{dx} = \sqrt{x - y}; \quad y(2) = 1$$
$$(b)\ \frac{dy}{dx} = \sqrt{x - y}; \quad y(1) = 2$$
$$(c)\ \frac{dy}{dx} = \sqrt{x - y}; \quad y(2) = 2$$

   (show your work.)

   - **ans:**

$$f = \sqrt{x - y}; \ \frac{\partial f}{\partial y} = -\frac{1}{2\sqrt{x - y}}$$

   (a) Both f and $\frac{\partial f}{\partial y}$ are continuous at a neighborhood of $(2, 1)$, so the theorem does guaranty a unique solution nearby.
   (b) f is not continuous at a neighborhood of $(1, 2)$ as we have $\sqrt{1 - 2}$, so the theorem does not guaranty anything about the solution.
   (c) f is not continuous at a neighborhood of $(2, 2)$, because when y is a little bigger than x we would have a negative square root, so the theorem does not guaranty anything about the solution.

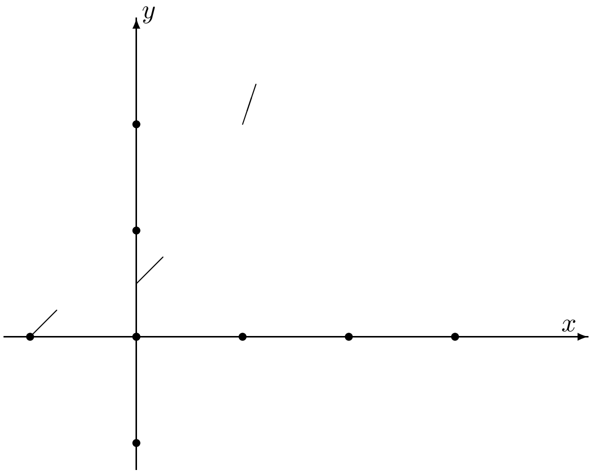
4. First construct a slope field (3 points are required: $(0, 1/2)$, $(1, 2)$, $(-1, 0)$.) Then sketch the solution curve corresponding to the given initial condition.

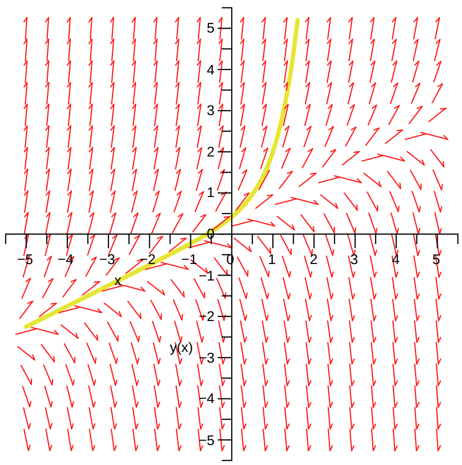
$$\frac{dy}{dx} = 2y - x; \quad y(0) = 1/2$$

   - **ans:**

(x, y)	$(0, 1/2)$	$(1, 2)$	$(-1, 0)$
$y' = 2y - x$	1	3	1

   We draw a little arrow at point $(0, 1/2)$ with a slope $y' = 1$.



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with(DEtools):
DEplot( diff(y(x),x)=2*y(x)-x,y(x),
  x=-5..5,[[y(0)=1/2]],y=-5..5,stepsize=.05);
```

5. Find the general solution of

$$\frac{dy}{dx} + 2xy^2 = 0$$

- **ans:**

$$\int \frac{dy}{y^2} = -\int 2x dx$$

$$-\frac{1}{y} = -x^2 - c$$

$$y = \frac{1}{x^2 + c}$$

When divided by y^2 above, we lose the singular solution $y = 0$, which is not included in the general solution.

$$y = 0$$

6. Solve

$$2xy' - 3y = 9x^2$$

- **ans:** Warning, we must make the coefficient of y' 1:

$$y' - \frac{3}{2x}y = \frac{9}{2}x$$

The integrating factor is

$$\rho = e^{\int -3/(2x)} = e^{-(3/2)\ln x} = x^{-3/2}$$

Multiply by the factor, the DE becomes

$$\frac{d}{dx}(x^{-3/2}y) = \frac{9}{2}x^{-1/2}$$

$$x^{-3/2}y = \int \frac{9}{2}x^{-1/2}dx$$
$$x^{-3/2}y = 9x^{1/2} + c;$$
$$y = 9x^2 + cx^{3/2}$$

```
dsolve({2x*(diff(y(x), x))-3*y(x)
   = 9x^2 }, y(x))
```

7. Solve

$$xy' - 3y = x^3, \; y(1) = 2$$

- **ans:**

$$y' + \frac{-3}{x}y = x^2$$

$$\rho = e^{\int p} = e^{-3\ln x} = x^{-3}$$

$$y = \rho^{-1}\int \rho q$$
$$= x^3 \int x^{-1} = x^3 \ln x + Cx^3$$

By $y(1) = 2$,

$$y = x^3 \ln x + 2x^3$$

```
dsolve({x*(diff(y(x), x))-3*y(x)
   = x^3, y(1) = 2}, y(x))
```

8. Solve the DE by regarding y as the independent variable.

$$y = (2y - x)\frac{dy}{dx}$$

- **ans:** Let $x' = dx/dy$.

$$yx' + x = 2y$$

$$x' + \frac{1}{y}x = 2$$

$$\rho = e^{\int p} = e^{\int (1/y)dy} = y$$

Multiply by the factor, the DE becomes

$$x = \rho^{-1}\int \rho q = \frac{1}{y}(y^2 + c)$$

9. Solve the differential equation as the type $y' = f(y/x)$:

$$2xyy' = 4x^2 + 3y^2$$

- **ans:** This is a homogeneous equation:

$$y' = 2\frac{x}{y} + \frac{3}{2}\frac{y}{x} = f(\frac{y}{x})$$

We always let

$$v = \frac{y}{x}$$

Then we always have

$$y = xv, \quad y' = v + xv'$$

so (divide the equation by x only)

$$v + xv' = \frac{2}{v} + \frac{3}{2}v$$

$$xv' = \frac{v^2 + 4}{2v}$$

$v = 0$ implies $y \equiv 0$, which is a solution.

$$\int \frac{2v}{v^2 + 4}dv = \int dx/x$$

$$v^2 + 4 = Cx$$

$$y^2 + 4x^2 = Cx^3$$

10. Show the differential equation is exact, and solve it as an exact equation.

$$(2\cos x + 2xy^2)dx + (2e^{2y} + 2x^2y)dy = 0$$

- **ans:** Exact equation: we write it in the form $F' = 0$:

$$Mdx + Ndy = 0$$

where $F = F(x, y)$, and x and y are independent variable. Then we must have

$$F_x = M, \ F_y = N, \quad M_y = N_x = F_{xy}$$

Here we check

$$M_y == 4xy, \ N_x = 4xy = M_y$$

Note, the above part can be omitted if you can do the second part below correctly. Because the second part, found a solution, implies the equation is exact!

How to find F?

$$F = \int Mdx = \int (2\cos x + 2xy^2)dx = 2\sin x + x^2y^2 + f(y)$$

$$F_y = 2x^2y + f'(y) = 2e^{2y} + 2x^2y, \quad f'(y) = 2e^{2y}$$

So $f = \int 2e^{2y}dy = e^{2y}$. Solution of $dF = 0$ is

$$F = C$$

$$2\sin x + x^2y^2 + e^{2y} = C$$

11. Reduce the equation to a first order one, and solve it.

$$y'' = (y')^3$$

- **ans:** No y! Let $z = y'$

$$z' = z^3$$

If $z = 0$, $y' = 0$, $y = C$. Yes, it is a solution.

If $z \neq 0$,

$$\frac{dz}{z^3} = dx$$

$$-\frac{1}{2z^2} = x + C$$

$$z^2 = \frac{1}{-2x - 2C}$$

$$y = \int \frac{1}{\sqrt{-2x - 2C}}dx = -\sqrt{-2x - 2C} + C_1$$

And

$$y = -\int \frac{1}{\sqrt{-2x - 2C}}dx = \sqrt{-2x - 2C} + C_2$$

12. Suppose that when a certain lake is stocked with fish, the birth and death rates β and δ are both proportional to $1/P^2$. If $P_0 = 900$ and after 6 months, there are 441 fish in the lake, how long did it take all the fish in the lake to die?

- **ans:** Let $k = (\beta - \delta)/P^2$.

$$\frac{d}{dt}P = k\frac{1}{P^2}P$$

$$\int PdP = kdt$$

$$P^2(t)/2 = kt + C,\ \ P = \sqrt{2kt + C_1}$$

$P(0) = 900$,P(6)=441,

$$441 = \sqrt{2k6 + 900^2}$$

then $k = -51293.25$, and

$$P(t) = \sqrt{-205173t + 810000}$$

$P = 0$, $t = 7.895$.

13. Suppose that a body is shot up through a resisting medium with resistance proportional to its velocity v. Find the height the body can reach if the gravity constant $g = 10$, mass $m = 1$, $v_0 = 100$ and $k = 0.1$.

- **ans:** Resistance proportional to velocity.

$$mv' = F_R + F_G = -kv - mg$$

$$v' + 0.1v = -10$$

It is a first order linear equation.

$$\rho = e^{\int p} = e^{0.1t}$$

$$v = \rho^{-1}\int \rho q = e^{-0.1t}(-100e^{0.1t} + C)$$

$v_0 = 100$ when $t = 0$.

$$C = 200$$

$$v = 200e^{-0.1t} - 100$$

$v = 0$, $t = \ln(1/2)/(-0.1) = 6.93$

$$y = \int v = -2000e^{-0.1t} - 100t + C$$

$$\begin{aligned} y_{\max} &= y(6.93) - y(0) \\ &= 306.85 \end{aligned}$$