M341 H11 (S. Zhang) L 6.1: 1(c-h), 3, 4, 6, 9
L 6.3: 1(a-b,d-e), 2(a-b), 8(a,b) .

80.20 1. (6.1:1c,) Find eigenvalues and eigenvectors.

$$\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

- **ans:**

$$\det(A - \lambda I) = 0,\ \lambda = 2, 2$$

$\lambda = 2$

$$A - \lambda I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},\ x = C \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We are lack of a second linearly independent eigenvector.

```
a=[3 -1; 1 1];
[p e]=eig(a)
```

80.21 2. (6.1:1d) Find eigenvalues and eigenvectors.

$$\begin{pmatrix} 3 & -8 \\ 2 & 3 \end{pmatrix}$$

- **ans:** (1)

$$\det(A - \lambda I) = 0,\ \lambda = 3 \pm 4i$$

$\lambda = 3 + 4i$

$$A - \lambda I = \begin{pmatrix} -4i & -8 \\ 2 & -4i \end{pmatrix},\ x = C \begin{pmatrix} 2 \\ -i \end{pmatrix}$$

We skip the other eigenvector as we know the eigenvalues are in conjugate pairs, and so are eigenvectors.

```
a=[3 -8; 2 3];
[p e]=eig(a)
```

80.22 3. (6.1:1e) Find eigenvalues and eigenvectors.

$$\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$$

- **ans:**

$$\det(A - \lambda I) = 0,\ \lambda = 2 \pm i$$

$\lambda = 2 + i$

$$A - \lambda I = \begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix},\ x = C \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

We skip the other eigenvector as we know the eigenvalues are in conjugate pairs, and so are eigenvectors.

```
a=[1 1; -2 3];
[p e]=eig(a)
```

80.23 4. (6.1:1f) Find eigenvalues and eigenvectors.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- **ans:** (2)

$$\det(A - \lambda I) = 0,\ \lambda = 0, 0, 0$$

$\lambda = 0$

$$A - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ x = C \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We have only 1 linearly independent eigenvector (lacking two, it is a defective matrix).

```
a=[0 1 0;0 0 1;0 0 0];
[p e]=eig(a)
```

80.24 5. (6.1:1g) Find eigenvalues and eigenvectors.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

- **ans:**

$$\det(A - \lambda I) = 0,\ \lambda = 1, 1, 2$$

$\lambda = 1$

$$A - \lambda I = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ x = C \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, C \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$\lambda = 2$

$$A - \lambda I = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix},\ x = C \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

```
a=[1 1 1;0 2 1;0 0 1];
[p e]=eig(a)
```

80.25 6. (6.1:1h,) Find eigenvalues and eigenvectors.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{pmatrix}$$

- **ans:**

$$\det(A - \lambda I) = 0,\ \lambda = 1, 4, -2$$

$\lambda = 1$

$$A - \lambda I = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 5 & -2 \end{pmatrix},\ x = C \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = 4$

$$A - \lambda I = \begin{pmatrix} -3 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \end{pmatrix}, \ x = C\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = -2$

$$A - \lambda I = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 5 & 1 \end{pmatrix}, \ x = C\begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix}$$

```
a=[1 2 1;0 3 1;0 5 -1];
[p e]=eig(a)
```

7. (6.1:3) Prove $A_{n\times n}$ is singular if and only if $\lambda = 0$ is
80.32 an eigenvalue of A.

- **ans:** Prove $\Rightarrow$.

If $A_{n\times n}$ is singular, then $\det A = 0$. Let $\lambda = 0$ in the polynomial

$$\det(A - \lambda I) = \det(A) = 0$$

so $\lambda = 0$ is a root of polynomial $\det(A - \lambda I) = 0$. Therefore, $\lambda = 0$ is an eigenvalue of A.

Prove $\Leftarrow$.

If $\lambda = 0$ is an eigenvalue of A, then $Ax = \lambda x = 0x = 0$ for some nonzero eigenvector x. Therefore $Ax = 0$ has nonzero solution and A is singular. (otherwise the only solution is $A^{-1}0 = 0$, the trivial solution.)

8. (6.1:4) Let A be a nonsingular matrix and let λ be an
80.34 eigenvalue of A. Show that $1/\lambda$ is an eigenvalue of A^{-1}.

- **ans:** Let $x \neq 0$ be a corresponding eigenvector, i.e., $Ax = \lambda x$. Multiplying both sides by A^{-1} and $1/\lambda$, we get

$$x = \lambda A^{-1}x$$
$$A^{-1}x\& = \frac{1}{\lambda}x$$

So $1/\lambda$ is an eigenvalue of A^{-1} as $x \neq 0$.

9. (6.1:6) An $n \times n$ matrix is said to be idempotent if
80.36 $A^2 = A$. Show that if λ is an eigenvalue of an idempotent matrix, then λ must be either 0 or 1.

- **ans:** Let $x \neq 0$ be a corresponding eigenvector, i.e., $Ax = \lambda x$. Multiplying both sides by A, we get

$$Ax = \lambda x$$
$$A^2x = \lambda Ax = \lambda^2 x$$

Now $A^2 = A$,

$$Ax = \lambda^2 x$$

Since $Ax = \lambda x$,

$$\lambda x = \lambda^2 x$$
$$\lambda(1 - \lambda)x = 0$$

As $x \neq 0$, we have either $\lambda = 0$ or $1 - \lambda = 0$.

10. (6.1:9) Show that A and A^T have the same eigenvalues.
80.38 Do they necessarily have the same eigenvectors? Explain.

- **ans:** If λ is an eigenvalue of A, then we have

$$\det(A - \lambda I) = 0$$

Now, the determinant is invariant under the transpose, i.e,

$$\det(A^T - \lambda I) = \det\left((A - \lambda I)^T\right) = \det(A - \lambda I) = 0.$$

So λ is an eigenvalue of A^T. By symmetry (between A and A^T, all eigenvalues of A^T are eigenvalues of A.

But the eigenvectors are completely different. For example,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \ \lambda = 1, 0, \ x_1 = C\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ x_2 = C\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

However,

$$A^T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \ \lambda = 1, 0, \ x_1 = C\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ x_2 = C\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

---

1. (6.3:1a) Factor the matrix A into a product XDX^{-1},
82.20 where D is diagonal.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- **ans:**

$$\det(A - \lambda I) = 0, \ \lambda = -1, 1$$

$\lambda = -1$

$$A - \lambda I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ x = C\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\lambda = 1$

$$A - \lambda I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \ x = C\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \ X^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$D = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

```
d=diag(eig([0 1;1 0]))
x=inv([1 1;-1 1])
inv(x)*d*x-a
```

2. (6.3:1b) Factor the matrix A into a product XDX^{-1},
82.22 where D is diagonal.

$$\begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$$

- **ans:**

$$\det(A-\lambda I)=0,\ \lambda=2,1$$

$\lambda=2$

$$A-\lambda I=\begin{pmatrix}3 & 6\\ -2 & -4\end{pmatrix},\ x=C\begin{pmatrix}2\\ -1\end{pmatrix}$$

$\lambda=1$

$$A-\lambda I=\begin{pmatrix}4 & 6\\ -2 & -3\end{pmatrix},\ x=C\begin{pmatrix}3\\ -2\end{pmatrix}$$

$$X=\begin{pmatrix}2 & 3\\ -1 & -2\end{pmatrix},\ X^{-1}=\begin{pmatrix}2 & 3\\ -1 & -2\end{pmatrix}$$
$$D=\begin{pmatrix}2 & \\ & 1\end{pmatrix}$$

```
a=[5 6;-2 -2]
d=diag(eig(a))
x=inv([2 3;-1 -2])
inv(x)*d*x-a
```

3. (6.3:1d) Factor the matrix A into a product XDX^{-1},
82.26 where D is diagonal.

$$\begin{pmatrix}2 & 2 & 1\\ 0 & 1 & 2\\ 0 & 0 & -1\end{pmatrix}$$

- **ans:**

$$\det(A-\lambda I)=0,\ \lambda=2,1,-1$$

$\lambda=2$

$$A-\lambda I=\begin{pmatrix}0 & 2 & 1\\ 0 & -1 & 2\\ 0 & 0 & -3\end{pmatrix},\ x=C\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

$\lambda=1$

$$A-\lambda I=\begin{pmatrix}1 & 2 & 1\\ 0 & 0 & 2\\ 0 & 0 & -2\end{pmatrix},\ x=C\begin{pmatrix}2\\ -1\\ 0\end{pmatrix}$$

$\lambda=-1$

$$A-\lambda I=\begin{pmatrix}3 & 2 & 1\\ 0 & 2 & 2\\ 0 & 0 & 0\end{pmatrix},\ x=C\begin{pmatrix}-1\\ 3\\ -3\end{pmatrix}$$

$$X=\begin{pmatrix}1 & 2 & -1\\ 0 & -1 & 3\\ 0 & 0 & -3\end{pmatrix},\ X^{-1}=\begin{pmatrix}1 & 2 & 5/3\\ 0 & -1 & -1\\ 0 & 0 & -1/3\end{pmatrix}$$
$$D=\begin{pmatrix}2 & & \\ & 1 & \\ & & -1\end{pmatrix}$$

```
a=[2 2 1;0 1 2; 0 0 -1]
d=diag(eig(a))
x=inv([1 2 -1;0 -1 3;0 0 -3])
inv(x)*d*x-a
```

4. (6.3:1e) Factor the matrix A into a product XDX^{-1},
82.28 where D is diagonal.

$$\begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 3\\ 1 & 1 & -1\end{pmatrix}$$

- **ans:**

$$\det(A-\lambda I)=0,\ \lambda=-2,2,1$$

$\lambda=-2$

$$A-\lambda I=\begin{pmatrix}3 & 0 & 0\\ -2 & 3 & 3\\ 1 & 1 & 1\end{pmatrix},\ x=C\begin{pmatrix}0\\ 1\\ -1\end{pmatrix}$$

$\lambda=2$

$$A-\lambda I=\begin{pmatrix}-1 & 0 & 0\\ -2 & -1 & 3\\ 1 & 1 & -3\end{pmatrix},\ x=C\begin{pmatrix}0\\ 3\\ 1\end{pmatrix}$$

$\lambda=1$

$$A-\lambda I=\begin{pmatrix}0 & 0 & 0\\ -2 & 0 & 3\\ 1 & 1 & -2\end{pmatrix},\ x=C\begin{pmatrix}3\\ 1\\ 2\end{pmatrix}$$

$$X=\begin{pmatrix}0 & 0 & 3\\ 1 & 3 & 1\\ -1 & 1 & 2\end{pmatrix},$$
$$X^{-1}=\begin{pmatrix}5/12 & 1/4 & -3/4\\ -1/4 & 1/4 & 1/4\\ 1/3 & 0 & 0\end{pmatrix}$$
$$D=\begin{pmatrix}-2 & & \\ & 2 & \\ & & 1\end{pmatrix}$$

```
a=[1 0 0;-2 1 3;1 1 -1]
d=diag(eig(a))
x=inv([0 0 3;1 3 1;-1 1 2]);
rats(x),inv(x)*d*x-a
```

5. Use $A=XDX^{-1}$ to find A^6.
82.32

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

- **ans:** 6.3:2a

$$\det(A-\lambda I)=0,\ \lambda=-1,1$$

$\lambda = -1$

$$A - \lambda I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ x = C\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\lambda = 1$

$$A - \lambda I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},\ x = C\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},\ X^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$D = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

$$\begin{aligned} A^6 =& XD^6X^{-1} = X\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}X^{-1} \\ =& XX^{-1} = I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \end{aligned}$$

```
d=diag(eig([0 1;1 0]))
x=inv([1 1;-1 1])
inv(x)*d*x
inv(x)*d^6*x
```

6. (6.3:2b) Factor the matrix A into a product XDX^{-1},
82.34 where D is diagonal.

$$\begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$$

- **ans:**

$$\det(A - \lambda I) = 0,\ \lambda = 2, 1$$

$\lambda = 2$

$$A - \lambda I = \begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix},\ x = C\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$\lambda = 1$

$$A - \lambda I = \begin{pmatrix} 4 & 6 \\ -2 & -3 \end{pmatrix},\ x = C\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$X = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix},\ X^{-1} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$$
$$D = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}$$

$$\begin{aligned} A^6 =& XD^6X^{-1} = X\begin{pmatrix} 2^6 & \\ & 1 \end{pmatrix}X^{-1} \\ =& \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 64 & \\ & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \\ =& \begin{pmatrix} 253 & 378 \\ -126 & -188 \end{pmatrix} \end{aligned}$$

```
a=[5 6;-2 -2]
d=diag(eig(a))
x=inv([2 3;-1 -2])
inv(x)*d*x-a
inv(x)*d^6*x
```

7. (6.3:8a) Find possible values for a that make the matrix
82.36 defective, or show that no such values exist.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$$

- **ans:**

$$\det(A - \lambda I) = 0,\ \lambda = 0, 2, a$$

If $a \neq 0, 2$, then we have 3 different eigenvalues, and the matrix would not be defective.

Let us check $a = 0$. We have repeated eigenvalue 0. $\lambda = 0$

$$A - \lambda I = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ x = C\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ C\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

We do have two linearly independent eigenvectors. So A is not defective.

Let us check $a = 2$. We have repeated eigenvalue 2. $\lambda = 2$

$$A - \lambda I = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ x = C\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ C\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We do have two linearly independent eigenvectors. So A is not defective.

Therefore, there is no such a.

```
a=2
A=[1 1 0; 1 1 0; 0 0 a];
[x e]=eig(A)
A*x, x*e
```

8. (6.3:8b) Find possible values for a that make the matrix
82.38 defective, or show that no such values exist.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & a \end{pmatrix}$$

- **ans:**

$$\det(A - \lambda I) = 0,\ \lambda = 0, 2, a$$

If $a \neq 0, 2$, then we have 3 different eigenvalues, and the matrix would not be defective.

Let us check $a = 0$. We have repeated eigenvalue 0. $\lambda = 0$

$$A - \lambda I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ x = C\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ C\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

We do have two linearly independent eigenvectors. So A is not defective.

Let us check $a = 2$. We have repeated eigenvalue 2. $\lambda = 2$

$$A - \lambda I = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ x = C \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

We have only one linearly independent eigenvector. So A is defective.

Therefore, only when $a = 2$, A is defective.

```
a=2
A=[1 1 1; 1 1 1; 0 0 a];
[x e]=eig(A)
A*x, x*e, x*e*inv(x)
```