M341 H10 (S. Zhang) 3.4: 2, 5, 8, 10, 14
L 3.5: 1, 2, 5, 6
L 3.6: 1, 2, 4, 5, 8.

1. (3.4:2) Determine whether the following vectors form
68.20 a basis in R^3.

(a) $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix} \right\}$

(c) $\left\{ \begin{pmatrix}2\\1\\-2\end{pmatrix}, \begin{pmatrix}3\\2\\-2\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix} \right\}$

(d) $\left\{ \begin{pmatrix}2\\1\\-2\end{pmatrix}, \begin{pmatrix}-2\\-1\\2\end{pmatrix}, \begin{pmatrix}4\\2\\-2\end{pmatrix} \right\}$

(e) $\left\{ \begin{pmatrix}1\\1\\3\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix} \right\}$

• **ans:** If the number of vectors is 3, we can compute the determinant. Otherwise they do not form a basis.

Note the difference between this problem, and the one finding a spanning set, or the one determining linear dependency.

(a) A basis. The determinant is 1 (non zero). So they are linearly independent and $Ax = 0$ has a unique solution 0. In addition, we have precise 3 vectors.

(b) Not a basis. Too many vectors (must be 3).

(c) Not a basis. The vectors are linearly dependent, because the determinant is zero (in this case, $Ax = 0$ has non-zero solutions):

$$\begin{vmatrix}2 & 3 & 2\\ 1 & 2 & 2\\ -2 & -2 & 0\end{vmatrix} = 0$$

(d) Not a basis. The vectors are linearly dependent, because the determinant is zero:

$$\begin{vmatrix}2 & -2 & 4\\ 1 & -1 & 2\\ -2 & 2 & 4\end{vmatrix} = 0$$

(e) Not a basis. Too few vectors (must be 3).

2. (3.4:5) Given
68.22

$$x_1 = \begin{pmatrix}2\\1\\3\end{pmatrix},\ x_2 = \begin{pmatrix}3\\-1\\4\end{pmatrix},\ x_3 = \begin{pmatrix}2\\6\\4\end{pmatrix}$$

(a) Show that x_1, x_2, x_3 are linearly dependent.

(b) Show that x_1, x_2 are linearly independent.

(c) What is the dimension of Span(x_1, x_2, x_3)?

(d) Give a geometric description of Span(x_1, x_2, x_3).

• **ans:**

(a) As we have precisely 3 vectors, we can simply check the determinant.

$$\begin{vmatrix}2 & 3 & 2\\ 1 & -1 & 6\\ 3 & 4 & 4\end{vmatrix}$$
$$=2(-1)4 - 2(6)4 + 3(6)3 - 3(1)(4) + 2(1)(4) - 2(-1)(3)$$
$$= -8 - 48 + 54 - 12 + 8 + 6 = 68 - 68 = 0$$

So the three vectors are linearly dependent.

(b) We solve the homogeneous system, $Ax = 0$.

$$\begin{pmatrix}2 & 3 & | & 0\\ 1 & -1 & | & 0\\ 3 & 4 & | & 0\end{pmatrix}$$
$$\to \begin{pmatrix}2 & 3 & | & 0\\ & -5/2 & | & 0\\ & -1/2 & | & 0\end{pmatrix}$$

So we have only zero solution $x = \begin{pmatrix}0\\0\end{pmatrix}$. So the two vectors are linearly independent.

(c) By (a), the dimension of Span(x_1, x_2, x_3) is less than 3. By (b), as we found two linearly independent vectors in Span(x_1, x_2, x_3), the dimension of Span(x_1, x_2, x_3) is at least 2. So the dimension of Span(x_1, x_2, x_3) is 2.

(d) By above work, we know the third vector is a linear combination of the first two vectors. The Span(x_1, x_2, x_3) =Span(x_1, x_2). All vectors in Span(x_1, x_2, x_3) are in the format

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = a\begin{pmatrix}2\\1\\3\end{pmatrix} + b\begin{pmatrix}3\\-1\\4\end{pmatrix}$$

for any real numbers a and b.

We can eliminate a and b by substituting the first and the second equation into the third equation. I.e., by adding -2 times the second equation to the first equation,

$$x - 2y = 5b$$

by adding 2 times the second equation to the first equation,

$$x + 3y = 5a$$

Put them into the third equation,

$$z = \frac{x + 3y}{5}(3) + \frac{x - 2y}{5}(4)$$

This is an equation for a plane in 3D, (must passing the origin).

$$7x + y - 5z = 0$$

(we can verify that the given three vectors all satisfy the plane equation.)

3. (3.4:8) Given

68.24

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; x_2 = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

(a) Do x_1 and x_2 span R^3?

(b) Let x_3 be a third vector in R^3 and set $X = (x_1 \; x_2 \; x_3)$. What condition(s) would X have to satisfy in order for $x_1, \; x_2, \; x_3$ to form a basis for R^3?

(c) Find a third vector x_3 that will extend the set $\{x_1, x_2\}$ to a basis for R^3.

- **ans:**

(a) No. $\dim R^3 = 3$. To span R^3, a set must have at least 3 (linearly independent) vectors.

(b) A simple condition is $\det X \neq 0$. In this case, the 3 column vectors of X are linearly independent. Any 3 linearly independent vectors would form a basis as $\dim R^3 = 3$.

(c) We can do it by a simple inspection. Let $x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Then

$$\det X = \begin{vmatrix} 1 & 3 & 0 \\ 1 & -1 & 0 \\ 1 & 4 & 1 \end{vmatrix} = -4 \neq 0$$

So the three form a basis.

In general, for example, in R^4 or R^5, it is not easy to do inspection. We can do it by adding a standard basis of R^3 to the set, then pare down the set to get a basis.

$$\begin{pmatrix} 1 & 3 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 1 \end{pmatrix}$$

$$\overset{\substack{-r_1+r_3 \\ -r_1+r_2}}{\longrightarrow} \begin{pmatrix} 1 & 3 & 1 & 0 & 0 \\ & -4 & -1 & 1 & 0 \\ & 1 & -1 & 0 & 1 \end{pmatrix}$$

$$\overset{r_2 \leftrightarrow r_3}{\longrightarrow} \begin{pmatrix} 1 & 3 & 1 & 0 & 0 \\ & 1 & -1 & 0 & 1 \\ & -4 & -1 & 1 & 0 \end{pmatrix}$$

$$\overset{4r_2+r_3}{\longrightarrow} \begin{pmatrix} 1 & 3 & 1 & 0 & 0 \\ & 1 & -1 & 0 & 1 \\ & & -5 & 1 & 4 \end{pmatrix}$$

We pick up the column vectors corresponding to the pivoting columns, so

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

form a basis.

4. (3.4:10) Given

68.26

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Pare down the set to form a basis for R^3.

- **ans:**

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 5 & 3 & 7 & 1 \\ 2 & 4 & 2 & 4 & 0 \end{pmatrix}$$

$$\overset{\substack{-2r_1+r_3 \\ -2r_1+r_2}}{\longrightarrow} \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ & 1 & 1 & 3 & -1 \\ & & & & -2 \end{pmatrix}$$

We pick up the column vectors corresponding to the pivoting columns (1, 2, 5), so

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

form a basis.

5. (3.4:14) In each of the following, find the dimension of

68.28 the subspace of P_2 (all degree 2 or less polynomials) spanned by the given vectors.

(a) $x, x-1, x^2+1$.

(b) $x, x-1, x^2+1, x^2-1$

(c) $x^2, x^2-x-1, x+1$

(d) $2x, x-2$

- **ans:** If we do all the work for determining a spanning set, we can easily answer the question of dimension, which is simply the number of vectors in a spanning set.

But this problem can be done easier by inspecting linearly independent vectors.

(a) $x, x-1, x^2+1$. The set is linearly independent (can check Wronskian). So the dimension of the spanned subspace is 3.

(b) $x, x-1, x^2+1, x^2-1$. As we show the first three are linearly independent, so the dimension of the spanned subspace is 3 (the maximum dimension.)

(c) $x^2, x^2-x-1, x+1$. It is obvious that the third vector is the first one subtracting the second one. So the third one is redundant in spanning. The dimension is 2.

(d) $2x, x-2$. The two vectors are linearly independent, so the dimension is 2.

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1. (3.5:1a,2a) Find the transition matrix from $[\mathbf{e}_1, \mathbf{e}_2]$ to

70.20 $[\mathbf{u}_1, \mathbf{u}_2]$, then the transition matrix from $[\mathbf{u}_1, \mathbf{u}_2]$ to $[\mathbf{e}_1, \mathbf{e}_2]$.

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; \mathbf{u}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

- **ans:** A given vector can be written as linear combinations of different basis.

$$v = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 = [\mathbf{e}_1, \mathbf{e}_2]\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$v = u_1\mathbf{u}_1 + u_2\mathbf{u}_2 = [\mathbf{u}_1, \mathbf{u}_2]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = U^{-1}I\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = S\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

In general, from V to U, $S = U^{-1}V$. From $[\mathbf{u}_1, \mathbf{u}_2]$ to $[\mathbf{e}_1, \mathbf{e}_2]$.

$$S = I^{-1}U = U = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

From $[\mathbf{e}_1, \mathbf{e}_2]$ to $[\mathbf{u}_1, \mathbf{u}_2]$,

$$S = U^{-1}I = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

```
u1=[1 1]'; u2=[-1 1]'
U=[u1 u2]
iU=inv(U)
```

2. (3.5:1b,2b) Find the transition matrix from $[\mathbf{e}_1, \mathbf{e}_2]$ to
70.22 $[\mathbf{u}_1, \mathbf{u}_2]$, then the transition matrix from $[\mathbf{u}_1, \mathbf{u}_2]$ to $[\mathbf{e}_1, \mathbf{e}_2]$.

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \mathbf{u}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

- **ans:** A given vector can be written as linear combinations of different basis.

$$v = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 = [\mathbf{e}_1, \mathbf{e}_2]\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$v = u_1\mathbf{u}_1 + u_2\mathbf{u}_2 = [\mathbf{u}_1, \mathbf{u}_2]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = U^{-1}I\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = S\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

In general, from V to U, $S = U^{-1}V$. From $[\mathbf{u}_1, \mathbf{u}_2]$ to $[\mathbf{e}_1, \mathbf{e}_2]$.

$$S = I^{-1}U = U = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$$

From $[\mathbf{e}_1, \mathbf{e}_2]$ to $[\mathbf{u}_1, \mathbf{u}_2]$,

$$S = U^{-1}I = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$$

```
u1=[1 2]'; u2=[2 5]'
U=[u1 u2]
iU=inv(U)
```

3. (3.5:1c,2c) Find the transition matrix from $[\mathbf{e}_1, \mathbf{e}_2]$ to
70.24 $[\mathbf{u}_1, \mathbf{u}_2]$, then the transition matrix from $[\mathbf{u}_1, \mathbf{u}_2]$ to $[\mathbf{e}_1, \mathbf{e}_2]$.

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \mathbf{u}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

- **ans:** A given vector can be written as linear combinations of different basis.

$$v = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 = [\mathbf{e}_1, \mathbf{e}_2]\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$
$$v = u_1\mathbf{u}_1 + u_2\mathbf{u}_2 = [\mathbf{u}_1, \mathbf{u}_2]\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = U^{-1}I\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = S\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

In general, from V to U, $S = U^{-1}V$. From $[\mathbf{u}_1, \mathbf{u}_2]$ to $[\mathbf{e}_1, \mathbf{e}_2]$.

$$S = I^{-1}U = U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

From $[\mathbf{e}_1, \mathbf{e}_2]$ to $[\mathbf{u}_1, \mathbf{u}_2]$,

$$S = U^{-1}I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

```
u1=[0 1]'; u2=[1 0]';
U=[u1 u2]
iU=inv(U)
```

4. (3.5:5) Let $\mathbf{u}_1 = (1,1,1)^T$, $\mathbf{u}_2 = (1,2,2)^T$, $\mathbf{u}_3 =$
70.26 $(2,3,4)^T$. Find the transition matrix corresponding to the change of basis from $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ to $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$. Find the coordinates of each of the following vectors with respect to $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$.

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

- **ans:** The transition matrix

$$S = U^{-1}I = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix},\ \mathbf{v}_u = Sv_v = S\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$$

To check:

$$Uv_u = V_v = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},\ \mathbf{v}_u = Sv_v = S\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix},\ \mathbf{v}_u = Sv_v = S\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

```
u1=[1 1 1]'; u2=[1 2 2]';
u3=[2 3 4]'; U=[u1 u2 u3]
iU=inv(U)
iU*[3 2 5; 1 1 2;2 3 4]'
```

5. (3.5:6) Let $\mathbf{u}_1 = (1,1,1)^T$, $\mathbf{u}_2 = (1,2,2)^T$, $\mathbf{u}_3 =$
70.28 $(2,3,4)^T$. Let $\mathbf{v}_1 = (4,6,7)^T$, $\mathbf{v}_2 = (0,1,1)^T$, $\mathbf{v}_3 = (0,1,2)^T$. Find the transition matrix from $[\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3]$ to $[\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3]$. Find the coordinates of $\mathbf{x} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3$ under the basis $[\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3]$

- **ans:** The transition matrix

$$\begin{aligned} S &= U^{-1}V \\ &= \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{v}_u &= S\mathbf{v}_v \\ &= \begin{pmatrix} 1 & -1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ 5 \\ -2 \end{pmatrix} \end{aligned}$$

To check:

$$U\mathbf{v}_u = V\mathbf{v}_v$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ 11 \\ 9 \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \\ 9 \end{pmatrix}$$

```
u1=[1 1 1]'; u2=[1 2 2]';
u3=[2 3 4]'; U=[u1 u2 u3]
u1=[4 6 7]'; u2=[0 1 1]';
u3=[0 1 2]'; V=[u1 u2 u3]
iU=inv(U)
uv=iU*V, v=[2 3 -4]'
uv*v
U*ans, V*v
```

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1. (3.6:1a) Find a basis for the row space, a basis for the
72.20 column space, and a basis for the nullspace.

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{pmatrix}$$

- **ans:** We reduce A to upper triangular matrix, and identify the rows and columns of pivots, which will tell the independent rows and columns of the original A.

$$\begin{aligned} \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{pmatrix} &\overset{-2r_1+r_2}{\longrightarrow} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 4 & 7 & 8 \end{pmatrix} \\ &\overset{-4r_1+r_3}{\longrightarrow} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & -5 & 0 \end{pmatrix} \\ &\overset{-r_2+r_3}{\longrightarrow} \begin{pmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Now it is easy to do backward substitution to find the solution of homogeneous system $Ax = 0$. These solutions x form the nullspace. Here, we have x_3 as a free variable.

$$N(A) = \text{Span}\{ \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \}$$

Note that we do not do row permutations here. So the original first rows are linearly independent. A basis for the row space of A is

$$\{ \begin{pmatrix} 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 4 \end{pmatrix} \}$$

A basis for the column space of A is

$$\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix} \}$$

```
A=[1 3 2;2 1 4;4 7 8]
A(2,:)=A(2,:)-2*A(1,:)
A(3,:)=A(3,:)-4*A(1,:)
A(3,:)=A(3,:)-1*A(2,:)
```

2. (3.6:1b) Find a basis for the row space, a basis for the
72.21 column space, and a basis for the nullspace.

$$A = \begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{pmatrix}$$

- **ans:** We reduce A to upper triangular matrix, and identify the rows and columns of pivots, which will tell the independent rows and columns of the original A. Not we try to avoid the row permutation.

$$\begin{pmatrix}-3 & 1 & 3 & 4\\ 1 & 2 & -1 & -2\\ -3 & 8 & 4 & 2\end{pmatrix} \overset{3r_1}{\longrightarrow} \begin{pmatrix}-3 & 1 & 3 & 4\\ 3 & 6 & -3 & -6\\ -3 & 8 & 4 & 2\end{pmatrix}$$

$$\overset{r_1+r_2}{\longrightarrow} \begin{pmatrix}-3 & 1 & 3 & 4\\ & 7 & 0 & -2\\ -3 & 8 & 4 & 2\end{pmatrix}$$

$$\overset{-r_1+r_3}{\longrightarrow} \begin{pmatrix}-3 & 1 & 3 & 4\\ & 7 & 0 & -2\\ & 7 & 1 & -2\end{pmatrix}$$

$$\overset{-r_2+r_3}{\longrightarrow} \begin{pmatrix}-3 & 1 & 3 & 4\\ & 7 & 0 & -2\\ & & 1 & 0\end{pmatrix}$$

Now it is easy to do backward substitution to find the solution of homogeneous system $Ax = 0$. These solutions x form the nullspace. Here, we have x_4 free.

$$N(A) = \text{Span}\{C \begin{pmatrix}10\\ 2\\ 0\\ 7\end{pmatrix}\},\ \dim N(A) = 1$$

A basis for the row space of A is

$$\{\begin{pmatrix}-3 & 1 & 3 & 4\end{pmatrix}, \begin{pmatrix}1 & 2 & -1 & -2\end{pmatrix}, \begin{pmatrix}-3 & 8 & 4 & 2\end{pmatrix}, \}$$

A basis for the column space of A is

$$\{\begin{pmatrix}-3\\ 1\\ -3\end{pmatrix}, \begin{pmatrix}1\\ 2\\ 8\end{pmatrix}, \begin{pmatrix}3\\ -1\\ 4\end{pmatrix}\}$$

```
A=[-3 1 3 4;1 2 -1 -2;-3 8 4 2]
A(2,:)=3*A(2,:)
A(2,:)=A(2,:)+1*A(1,:)
A(3,:)=A(3,:)-1*A(1,:)
A(3,:)=A(3,:)-1*A(2,:)
```

3. (3.6:1c) Find a basis for the row space, a basis for the
72.22 column space, and a basis for the nullspace.

$$A = \begin{pmatrix}1 & 3 & -2 & 1\\ 2 & 1 & 3 & 2\\ 3 & 4 & 5 & 6\end{pmatrix}$$

• **ans:** We reduce A to upper triangular matrix, and identify the rows and columns of pivots, which will tell the independent rows and columns of the original A. Not we try to avoid the row permutation.

$$\begin{pmatrix}1 & 3 & -2 & 1\\ 2 & 1 & 3 & 2\\ 3 & 4 & 5 & 6\end{pmatrix} \overset{-2r_1+r_2}{\longrightarrow} \begin{pmatrix}1 & 3 & -2 & 1\\ & -5 & 7 & 0\\ 3 & 4 & 5 & 6\end{pmatrix}$$

$$\overset{-3r_1+r_3}{\longrightarrow} \begin{pmatrix}1 & 3 & -2 & 1\\ & -5 & 7 & 0\\ & -5 & 11 & 3\end{pmatrix}$$

$$\overset{-r_2+r_3}{\longrightarrow} \begin{pmatrix}1 & 3 & -2 & 1\\ & -5 & 7 & 0\\ & & 4 & 3\end{pmatrix}$$

Now it is easy to do backward substitution to find the solution of homogeneous system $Ax = 0$. These solutions x form the nullspace. Here, we have x_4 free. Let $x_4 = C$, then

$$x = \begin{pmatrix}13/20\\ -21/20\\ -3/4\\ 1\end{pmatrix} C$$

$$N(A) = \text{Span}\{\begin{pmatrix}13\\ -21\\ -15\\ 20\end{pmatrix}\},\ \dim N(A) = 1$$

A basis for the row space of A is

$$\{\begin{pmatrix}1 & 3 & -2 & 1\end{pmatrix}, \begin{pmatrix}2 & 1 & 3 & 2\end{pmatrix}, \begin{pmatrix}3 & 4 & 5 & 6\end{pmatrix}, \}$$

A basis for the column space of A is

$$\{\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}, \begin{pmatrix}3\\ 1\\ 4\end{pmatrix}, \begin{pmatrix}-2\\ 3\\ 5\end{pmatrix}\}$$

```
A=[1 3 -2 1;2 1 3 2;3 4 5 6]
A(2,:)=A(2,:)-2*A(1,:)
A(3,:)=A(3,:)-3*A(1,:)
A(3,:)=A(3,:)-1*A(2,:)
```

4. (3.6:2a) Determine the dimension of the subspace of
72.24 R^3 spanned by the given vectors.

$$\mathbf{v}_1 = \begin{pmatrix}1\\ -2\\ 2\end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix}2\\ -2\\ 4\end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix}-3\\ 3\\ 6\end{pmatrix}$$

• **ans:** We write the vectors together as a matrix $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$. We then do row operations as if we find a basis for the column space of A.

$$A = \begin{pmatrix}1 & 2 & -3\\ -2 & -2 & 3\\ 2 & 4 & 6\end{pmatrix} \overset{2r_1+r_2}{\longrightarrow} \begin{pmatrix}1 & 2 & -3\\ & 2 & -3\\ 2 & 4 & 6\end{pmatrix}$$

$$\overset{-2r_1+r_3}{\longrightarrow} \begin{pmatrix}1 & 2 & -3\\ & 2 & -3\\ & 0 & 12\end{pmatrix}$$

Therefore, the dimension is 3. (The dimension of $N(A)$ is 0.)

We note that in this case, we can compute $\det A = 24$ so the three vectors are linearly independent. So they span the whole R^3 space, and the dimension of span is 3.

```
A=[1 -2 2;2 -2 4;-3 3 6]'
A(2,:)=A(2,:)+2*A(1,:)
A(3,:)=A(3,:)-2*A(1,:)
```

5. (3.6:2b) Determine the dimension of the subspace of
72.25 R^3 spanned by the given vectors.

$$\mathbf{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix}1\\2\\3\end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix}2\\3\\1\end{pmatrix}$$

- **ans:** We write the vectors together as a matrix $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$. We then do row operations as if we find a basis for the column space of A.

$$A = \begin{pmatrix}1 & 1 & 2\\1 & 2 & 3\\1 & 3 & 1\end{pmatrix} \overset{-r_1+r_2}{\longrightarrow} \begin{pmatrix}1 & 1 & 2\\ & 1 & 1\\1 & 3 & 1\end{pmatrix}$$
$$\overset{-r_1+r_3}{\longrightarrow} \begin{pmatrix}1 & 1 & 2\\ & 1 & 1\\ & 2 & -1\end{pmatrix}$$
$$\overset{-2r_2+r_3}{\longrightarrow} \begin{pmatrix}1 & 1 & 2\\ & 1 & 1\\ & & -3\end{pmatrix}$$

Therefore, the dimension is 3. (The dimension of $N(A)$ is 0.)

We note that in this case, we can compute $\det A = -3$ so the three vectors are linearly independent. So they span the whole R^3 space, and the dimension of span is 3.

```
A=[1 1 1;1 2 3; 2 3 1]'
A(2,:)=A(2,:)-1*A(1,:)
A(3,:)=A(3,:)-1*A(1,:)
A(3,:)=A(3,:)-2*A(2,:)
```

6. (3.6:2c) Determine the dimension of the subspace of
72.26 R^3 spanned by the given vectors.

$$\mathbf{v}_1 = \begin{pmatrix}1\\-1\\2\end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix}-2\\2\\-4\end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix}3\\-2\\5\end{pmatrix},\ \mathbf{v}_4 = \begin{pmatrix}2\\-1\\3\end{pmatrix}$$

- **ans:** We write the vectors together as a matrix $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$. We then do row operations as if we find a basis for the column space of A.

$$A = \begin{pmatrix}1 & -2 & 3 & 2\\-1 & 2 & -2 & -1\\2 & -4 & 5 & 3\end{pmatrix} \overset{r_1+r_2}{\longrightarrow} \begin{pmatrix}1 & -2 & 3 & 2\\ & 0 & 1 & 1\\2 & -4 & 5 & 3\end{pmatrix}$$
$$\overset{-2r_1+r_3}{\longrightarrow} \begin{pmatrix}1 & -2 & 3 & 2\\ & 0 & 1 & 1\\ & 0 & -1 & -1\end{pmatrix}$$
$$\overset{r_2+r_3}{\longrightarrow} \begin{pmatrix}1 & -2 & 3 & 2\\ & 0 & 1 & 1\\ & & & 0\end{pmatrix}$$

Therefore, the dimension is 2. (The dimension of $N(A)$ is 2.)

By the way, a basis for the Span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ is

$$\mathbf{v}_1 = \begin{pmatrix}1\\-1\\2\end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix}3\\-2\\5\end{pmatrix}$$

```
A=[1 -1 2;-2 2 -4;3 -2 5;2 -1 3]'
A(2,:)=A(2,:)+1*A(1,:)
A(3,:)=A(3,:)-2*A(1,:)
A(3,:)=A(3,:)+1*A(2,:)
```

7. (3.6:45ab) Determine if $\mathbf{b}$ is in the column space of A
72.28 and state whether the system $A\mathbf{x} = \mathbf{b}$ is consistent. If the system is consistent, determine whether there will be one or infinitely many solutions.

$$(a)\ A = \begin{pmatrix}1 & 2\\2 & 4\end{pmatrix},\ \mathbf{b} = \begin{pmatrix}4\\8\end{pmatrix}$$
$$(b)\ A = \begin{pmatrix}3 & 6\\1 & 2\end{pmatrix},\ \mathbf{b} = \begin{pmatrix}1\\1\end{pmatrix}$$

- **ans:** We need to solve the linear system $[A|b]$. (a)

$$\begin{pmatrix}1 & 2 & | & 4\\2 & 4 & | & 8\end{pmatrix} \overset{-2r_1+r_2}{\longrightarrow} \begin{pmatrix}1 & 2 & | & 4\\ & 0 & | & 0\end{pmatrix}$$

We can find solutions. So the system is consistent and $\mathbf{b}$ is in the column space of A. And there are infinitely many solutions.

(b)

$$\begin{pmatrix}3 & 6 & | & 1\\1 & 2 & | & 1\end{pmatrix} \overset{(-1/3)r_1+r_2}{\longrightarrow} \begin{pmatrix}3 & 6 & | & 1\\ & 0 & 2/3 & \end{pmatrix}$$

There is no solution. So the system is inconsistent and $\mathbf{b}$ is NOT in the column space of A.

```
a=[1 2;2 4]; b=[4 8]';
A=[a b]
A(2,:)=A(2,:)-2*A(1,:)
a=[3 6;1 2]; b=[1 1]';
A=[a b]
A(2,:)=A(2,:)-1/3*A(1,:)
```

8. (3.6:45c) Determine if **b** is in the column space of A
72.29 and state whether the system $A\mathbf{x} = \mathbf{b}$ is consistent. If the system is consistent, determine whether there will be one or infinitely many solutions.

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

- **ans:** We need to solve the linear system $[A|b]$.

$$\begin{pmatrix} 2 & 1 & | & 4 \\ 3 & 4 & | & 6 \end{pmatrix} \overset{(-3/2)r_1+r_2}{\longrightarrow} \begin{pmatrix} 2 & 1 & | & 4 \\ & 5/2 & | & 0 \end{pmatrix}$$

We can find a solution. So the system is consistent and **b** is in the column space of A. And there is a unique solution, $(2,0)^T$.

```
a=[2 1;3 4]; b=[4 6]';
A=[a b]
A(2,:)=A(2,:)-3/2*A(1,:)
```

9. (3.6:45d) Determine if **b** is in the column space of A
72.30 and state whether the system $A\mathbf{x} = \mathbf{b}$ is consistent. If the system is consistent, determine whether there will be one or infinitely many solutions.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

- **ans:** We need to solve the linear system $[A|b]$.

$$\begin{pmatrix} 1 & 1 & 2 & | & 1 \\ 1 & 1 & 2 & | & 2 \\ 1 & 1 & 2 & | & 3 \end{pmatrix} \overset{-r_1+r_2}{\longrightarrow} \begin{pmatrix} 1 & 1 & 2 & | & 1 \\ & 0 & 0 & | & 1 \\ 1 & 1 & 2 & | & 3 \end{pmatrix}$$

$$\overset{-r_1+r_3}{\longrightarrow} \begin{pmatrix} 1 & 1 & 2 & | & 1 \\ & 0 & 0 & | & 1 \\ & 0 & 0 & | & 2 \end{pmatrix}$$

There is no solution. So the system is inconsistent and **b** is NOT in the column space of A.

```
a=[1 1 2;1 1 2; 1 1 2]; b=[1 2 3]';
A=[a b]
A(2,:)=A(2,:)-1*A(1,:)
A(3,:)=A(3,:)-1*A(1,:)
```

10. (3.6:45e) Determine if **b** is in the column space of A
72.31 and state whether the system $A\mathbf{x} = \mathbf{b}$ is consistent. If the system is consistent, determine whether there will be one or infinitely many solutions.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$$

- **ans:** We need to solve the linear system $[A|b]$.

$$\begin{pmatrix} 0 & 1 & | & 2 \\ 1 & 0 & | & 5 \\ 0 & 1 & | & 2 \end{pmatrix}$$

It is easy to see there is a solution

$$x = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

So the system is consistent and **b** is in the column space of A. The solution is unique.

```
a=[ ]; b=[ ]';
A=[a b]
A(2,:)=A(2,:)-1*A(1,:)
A(3,:)=A(3,:)-1*A(1,:)
```

11. (3.6:45f) Determine if **b** is in the column space of A
72.32 and state whether the system $A\mathbf{x} = \mathbf{b}$ is consistent. If the system is consistent, determine whether there will be one or infinitely many solutions.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 5 \\ 10 \\ 5 \end{pmatrix}$$

- **ans:** We need to solve the linear system $[A|b]$.

$$\begin{pmatrix} 1 & 2 & | & 5 \\ 2 & 4 & | & 10 \\ 1 & 2 & | & 5 \end{pmatrix} \overset{-2r_1+r_2}{\longrightarrow} \begin{pmatrix} 1 & 2 & | & 5 \\ & 0 & | & 0 \\ 1 & 2 & | & 5 \end{pmatrix}$$

$$\overset{-r_1+r_3}{\longrightarrow} \begin{pmatrix} 1 & 2 & | & 5 \\ & 0 & | & 0 \\ & 0 & | & 0 \end{pmatrix}$$

It is easy to see there is a solution

$$x = \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$

So the system is consistent and **b** is in the column space of A. However, the rank of matrix A is 1. We have infinitely many solutions. All solutions can be written as

$$x = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + C \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

```
a=[1 2;2 4;1 2 ]; b=[5 10 5 ]';
A=[a b]
A(2,:)=A(2,:)-2*A(1,:)
A(3,:)=A(3,:)-1*A(1,:)
```

12. (3.6:8) Let A and B be 6×5 matrices. If $\dim N(A) = 2$,
72.34 what is the rank of A? If the rank of B is 4, what is the dimension of $N(B)$?

- **ans:** It holds that

$$\text{rank}(A) + \dim N(A) = n \text{ (the number of columns in } A \text{)}$$

So If $\dim N(A) = 2$, rank(A)=3. If the rank of B is 4, $\dim N(B) = 1$.