M341 H8 (S. Zhang) 1.4,2.1.

1. (1.4:3) For each of the following pairs of matrices, find
56.20 an elementary matrix E such that $EA = B$.

$$(a)\ A = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix},\ B = \begin{pmatrix} -4 & 2 \\ 5 & 3 \end{pmatrix}$$

$$(b)\ A = \begin{pmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 3 & 1 & 4 \end{pmatrix},\ B = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ -2 & 4 & 5 \end{pmatrix}$$

$$(c)\ A = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ -2 & 3 & 1 \end{pmatrix},\ B = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ 0 & 3 & 5 \end{pmatrix}$$

- **ans:** (a)

$$E = \begin{pmatrix} -2 & \\ & 1 \end{pmatrix}$$

(b)

$$E = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & \end{pmatrix}$$

(c)

$$E = \begin{pmatrix} 1 & & \\ & 1 & \\ & 2 & 1 \end{pmatrix}$$

```
    A=[2  -1 ;5  3];
     B =[-4   2  ;5   3];
      E=B*inv(A)
    A=[2 1 3  ; -2 4 5 ; 3 1 4 ];
    B=[2 1 3  ; 3 1 4  ; -2 4 5] ;
    E=B*inv(A)
A=[4 -2 3 ;1 0 2 ;-2 3 1];
B =[4 -2 3 ;1 0 2 ;0 3 5];
E=B*inv(A)
```

2. (1.4:6) Given
56.22

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{pmatrix}$$

Find elementary matrices such that

$$E_3E_2E_1A = U$$

Determine the inverses of the elementary matrices and set

$$L = E_1^{-1}E_2^{-1}E_3^{-1}$$

What type of matrix is L? Verify $A = LU$.

- **ans:**

$$\begin{pmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{pmatrix} \xrightarrow[-2r_1+r_3]{-3r_1+r_2} \begin{pmatrix} 2 & 1 & 1 \\ & 1 & 2 \\ & -1 & 1 \end{pmatrix} \xrightarrow{r_2+r_3} \begin{pmatrix} 2 & 1 & 1 \\ & 1 & 2 \\ & & 3 \end{pmatrix} = U$$

$$E_1 = \begin{pmatrix} 1 & & \\ -3 & 1 & \\ & & 1 \end{pmatrix},\ E_1^{-1} = \begin{pmatrix} 1 & & \\ 3 & 1 & \\ & & 1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ -2 & & 1 \end{pmatrix},\ E_2^{-1} = \begin{pmatrix} 1 & & \\ & 1 & \\ 2 & & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{pmatrix},\ E_3^{-1} = \begin{pmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{pmatrix}$$

$$L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{pmatrix} 1 & & \\ 3 & 1 & \\ 2 & -1 & 1 \end{pmatrix}$$

$$LU = A = \begin{pmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{pmatrix}$$

```
A=[2 1 1; 6 4 5; 4 1 3];AA=A;
A(2,:)=A(2,:)-3*A(1,:);
A(3,:)=A(3,:)-2*A(1,:),
A(3,:)=A(3,:)+A(2,:), U=A; I=eye(3);
B=A;
A=I; A(2,:)=A(2,:)-3*A(1,:);
iE1=inv(A), E1=A
A=I; A(3,:)=A(3,:)-2*A(1,:);
iE2=inv(A), E2=A,
A=I; A(3,:)=A(3,:)+A(2,:);
iE3=inv(A), E3=A,
L= iE1* iE2* iE3
AA-L*U
```

3. (1.4:9) Given
56.24

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}$$

Verify that

$$A^{-1} = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & 1 \\ 0 & -2 & 3 \end{pmatrix}$$

Use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$

$$\mathbf{b} = \begin{pmatrix}1\\1\\1\end{pmatrix};\ \begin{pmatrix}1\\2\\3\end{pmatrix},\ \begin{pmatrix}-2\\1\\0\end{pmatrix}$$

- **ans:**

$$AA^{-1} = \begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix}$$

$$A^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\-1\\1\end{pmatrix}$$

$$A^{-1}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}-4\\-2\\5\end{pmatrix}$$

$$A^{-1}\begin{pmatrix}-2\\1\\0\end{pmatrix} = \begin{pmatrix}0\\3\\-2\end{pmatrix}$$

```
A=[1 0 1; 3 3 4; 2 2 3];inv(A)
inv(A)*[1;1;1], inv(A)*[1;2;3],
 inv(A)*[-2;1;0],
```

4. (1.4:10af) Find A^{-1}.

56.26

$$A = \begin{pmatrix}-1 & 1\\ 1 & 0\end{pmatrix}$$

$$A = \begin{pmatrix}2 & 0 & 5\\ 0 & 3 & 0\\ 1 & 0 & 3\end{pmatrix}$$

- **ans:**

$$(A\ I) = \begin{pmatrix}-1 & 1 & 1 & 0\\ 1 & 0 & 0 & 1\end{pmatrix}$$
$$\overset{r_1\leftrightarrow r_2}{\longrightarrow}\begin{pmatrix}1 & 0 & 0 & 1\\ -1 & 1 & 1 & 0\end{pmatrix}$$
$$\overset{r_1+r_2}{\longrightarrow}\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 1\end{pmatrix} = (I\ A^{-1})$$

$$A^{-1} = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$$

$$(A\ I) = \begin{pmatrix}2 & 0 & 5 & 1 & 0 & 0\\ 0 & 3 & 0 & 0 & 1 & 0\\ 1 & 0 & 3 & 0 & 0 & 1\end{pmatrix}$$
$$\overset{r_1\leftrightarrow r_3}{\longrightarrow}\begin{pmatrix}1 & 0 & 3 & 0 & 0 & 1\\ 0 & 3 & 0 & 0 & 1 & 0\\ 2 & 0 & 5 & 1 & 0 & 0\end{pmatrix}$$
$$\overset{-2r_1+r_3}{\longrightarrow}\begin{pmatrix}1 & 0 & 3 & 0 & 0 & 1\\ 0 & 3 & 0 & 0 & 1 & 0\\ 0 & 0 & -1 & 1 & 0 & -2\end{pmatrix}$$
$$\overset{\substack{(1/3)r_2\\ r_3+r_1(\to r_1)}}{\longrightarrow}\begin{pmatrix}1 & 0 & 0 & 3 & 0 & -5\\ 0 & 1 & 0 & 0 & 1/3 & 0\\ 0 & 0 & -1 & 1 & 0 & -2\end{pmatrix}$$
$$\overset{(-1)r_3}{\longrightarrow}\begin{pmatrix}1 & 0 & 0 & 3 & 0 & -5\\ 0 & 1 & 0 & 0 & 1/3 & 0\\ 0 & 0 & 1 & -1 & 0 & 2\end{pmatrix}$$

$$A^{-1} = \begin{pmatrix}3 & 0 & -5\\ 0 & 1/3 & 0\\ -1 & 0 & 2\end{pmatrix}$$

```
A=[-1 1 1 0; 1 0 0 1];D=A(:,1:2);
B=A(1,:); A(1,:)=A(2,:);A(2,:)=B
A(2,:)=A(2,:)+A(1,:);
C=A(:,3:4), D*C
```

```
D=[2 0 5; 0 3 0; 1 0 3];A=[D eye(3)],
B=A(1,:); A(1,:)=A(3,:);A(3,:)=B
A(3,:)=A(3,:)-2*A(1,:)
A(2,:)=A(2,:)/3;
A(1,:)=A(1,:)+3*A(3,:)
A(3,:)=A(3,:)*(-1)
C=A(:,4:6), D*C
```

5. (1.4:10be) Find A^{-1}.

56.44

$$A = \begin{pmatrix}2 & 5\\ 1 & 3\end{pmatrix}$$

$$A = \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$$

- **ans:**

$$(A\ I) = \begin{pmatrix}2 & 5\\ 1 & 3\end{pmatrix}, \begin{pmatrix}1 & \\ 0 & 1\end{pmatrix}$$
$$\overset{(-1/2)r_1+r_2}{\longrightarrow}\begin{pmatrix}2 & 5\\ & 0.5\end{pmatrix}, \begin{pmatrix}1 & \\ -0.5 & 1\end{pmatrix}$$
$$\overset{2r_2;(1/2)r_1}{\longrightarrow}\begin{pmatrix}1 & 2.5\\ & 1\end{pmatrix}, \begin{pmatrix}0.5 & \\ -1 & 2\end{pmatrix}$$
$$\overset{(-2.5)r_2+r_1}{\longrightarrow}\begin{pmatrix}1 & \\ & 1\end{pmatrix}, \begin{pmatrix}3 & -5\\ -1 & 2\end{pmatrix}$$
$$= (I\ A^{-1})$$

$$A^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

$$\begin{aligned}
(A\ I) &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ 0 & 1 & \\ & & 1 \end{pmatrix} \\
&\overset{-r2+r_1}{\longrightarrow} \begin{pmatrix} 1 & & \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & \\ 0 & 1 & \\ & & 1 \end{pmatrix} \\
&\overset{-r_3+r_2}{\longrightarrow} \begin{pmatrix} 1 & & \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & \\ 0 & 1 & -1 \\ & & 1 \end{pmatrix} \\
&= (I\ A^{-1})
\end{aligned}$$

$$A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

```
A=[[2 5;1 3] eye(2,2)];
A(2,:)=(-1/2)*A(1,:)+A(2,:),
A(2,:)=2*A(2,:),
A(1,:)=(1/2)*A(1,:),
A(1,:)=(-2.5)*A(2,:)+A(1,:),
```

```
A=[[1 1 1;0 1 1;0 0 1] eye(3,3)];
A(1,:)=(-1)*A(2,:)+A(1,:),
A(2,:)=(-1)*A(3,:)+A(2,:),
```

6. (1.4:14) Let U and R be $n \times n$ upper triangular matrices
56.28 and set $T = UR$. Show that T is also upper triangular and

$$t_{jj} = u_{jj} r_{jj},\ j = 1, ..., n$$

- **ans:**

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & & \ddots & \\ & & & u_{nn} \end{pmatrix}$$

$$R = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & \cdots & r_{2n} \\ & & \ddots & \\ & & & r_{nn} \end{pmatrix}$$

$$T = \begin{pmatrix} u_{11}r_{11} & u_{11}r_{12} + u_{12}r_{22} & \cdots & \sum u_{1i}r_{in} \\ & u_{22}r_{22} & \cdots & \sum u_{2i}r_{in} \\ & & \ddots & \\ & & & u_{nn}r_{nn} \end{pmatrix}$$

7. (1.4:18) Let A and B be $n \times n$ matrices and let $C = AB$.
56.30 Prove that if B is singular then C must be singular.

- **ans:** If B is singular, there is at least one nonzero solution $\mathbf{x}$ to $B\mathbf{x} = \mathbf{0}$. Therefore, we found a nonzero solution $\mathbf{x}$ to $C\mathbf{x} = \mathbf{0}$, because

$$C\mathbf{x} = AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$$

Therefore C is singular.

---

1. (2.1:2) Use determinants to determine whether the fol-
60.20 lowing 2×2 matrices are nonsingular.

$$\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix},\ \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix},\ \begin{pmatrix} 3 & -6 \\ 2 & 4 \end{pmatrix}$$

- **ans:** If the determinant is nonzero, it is nonsingular.

$$\det \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix} = 2 \neq 0, \quad \text{nonsingular}$$
$$\det \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} = 0, \quad \text{singular}$$
$$\det \begin{pmatrix} 3 & -6 \\ 2 & 4 \end{pmatrix} = 24 \neq 0, \quad \text{nonsingular}$$

2. (2.1:3) Find $\det A$.
60.22

$$\begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & -2 \\ 2 & 1 & 3 \end{pmatrix},\ \begin{pmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{pmatrix},\ \begin{pmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ -3 & 2 & 3 & 1 \end{pmatrix}$$

- **ans:** Expansion by the first row.

$$\begin{aligned}
&\det \begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & -2 \\ 2 & 1 & 3 \end{pmatrix} \\
=&\det \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} - 3\det \begin{pmatrix} 4 & -2 \\ 2 & 3 \end{pmatrix} + 2\det \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} \\
=&5 - 3(16) + 2(2) = -39
\end{aligned}$$

Expansion by the third row.

$$\begin{aligned}
&\det \begin{pmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{pmatrix} \\
=&5\det \begin{pmatrix} -1 & 2 \\ 3 & 2 \end{pmatrix} - \det \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} + 6\det \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \\
=&5(-8) - (2) + 6(7) = 0
\end{aligned}$$

Expansion by row 2.

$$\begin{aligned}
&\det\begin{pmatrix}2&0&0&1\\0&1&0&0\\1&6&2&0\\1&1&-2&3\end{pmatrix}\\
=&\det\begin{pmatrix}2&0&1\\1&2&0\\1&-2&3\end{pmatrix}\\
=&2\det\begin{pmatrix}2&0\\-2&3\end{pmatrix}+\det\begin{pmatrix}1&2\\1&-2\end{pmatrix}\\
=&2(2)(3)+(-2-2)=8
\end{aligned}$$

Expansion by row 2.

$$\begin{aligned}
&\det\begin{pmatrix}2&1&2&1\\3&0&1&1\\-1&2&-2&1\\-3&2&3&1\end{pmatrix}\\
=&-3\det\begin{pmatrix}1&2&1\\2&-2&1\\2&3&1\end{pmatrix}\\
&-\det\begin{pmatrix}2&1&1\\-1&2&1\\-3&2&1\end{pmatrix}+\det\begin{pmatrix}2&1&2\\-1&2&-2\\-3&2&3\end{pmatrix}\\
=&20
\end{aligned}$$

```
det([1 3 2; 4 1 -2;2 1 3])
det([2 -1 2;1 3 2; 5 1 6])
det([2 0 0 1;0 1 0 0;1 6 2 0;1 1 -2 3])
det([2 1 2 1;3 0 1 1;-1 2 -2 1;-3 2 3 1])
```

3. (2.1:4) Find $\det A$ by inspection.

60.24

$$\begin{pmatrix}3&5\\2&4\end{pmatrix},\ \begin{pmatrix}2&0&0\\4&1&0\\7&3&-2\end{pmatrix}$$

$$\begin{pmatrix}3&0&0\\2&1&1\\1&2&2\end{pmatrix},\ \begin{pmatrix}4&0&2&1\\5&0&4&2\\2&0&3&4\\1&0&2&3\end{pmatrix}$$

- **ans:**

$$\det\begin{pmatrix}3&5\\2&4\end{pmatrix}=12-10=2$$

Expanding sequentially, row 1, row 2, ..., the determinant of a triangular matrix is the product of the diagonal elements.

$$\det\begin{pmatrix}2&0&0\\4&1&0\\7&3&-2\end{pmatrix}=2(1)(-2)=4$$

After expanding the first row, the remaining 2×2 matrix has two rows multiple of one another. So $\det=0$.

$$\det\begin{pmatrix}3&0&0\\2&1&1\\1&2&2\end{pmatrix}=\det\begin{pmatrix}1&1\\2&2\end{pmatrix}=0$$

Expanding by column 2, all terms are zero:

$$\det\begin{pmatrix}4&0&2&1\\5&0&4&2\\2&0&3&4\\1&0&2&3\end{pmatrix}=0$$