M341 H7 (S. Zhang) 1.2 1.3.

1. (1.2:1(a-f),) Determine if it is in row echelon form, or
52.20 reduced row echelon form.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} - & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

- **ans:**

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

The above matrix is in REF, but not RREF.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The above matrix is not in REF, (of course) not RREF.

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The above matrix is in REF, and is in RREF.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The above matrix is in REF, and is in RREF.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

The above matrix is not in REF (last row entry 3, must be 1), (of course)not RREF.

$$\begin{pmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

The above matrix is not in REF , (of course) not RREF.

2. (1.2:2(a-c),) Find the consistence. For those having a
52.22 unique solution, find the solution too.

$$\begin{pmatrix} 1 & 2 & | & 4 \\ 0 & 1 & | & 3 \\ 0 & 0 & | & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & | & 1 \\ 0 & 1 & | & -1 \\ 0 & 0 & | & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 4 & | & 1 \\ 0 & 0 & 1 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

- **ans:**

$$\begin{pmatrix} 1 & 2 & | & 4 \\ 0 & 1 & | & 3 \\ 0 & 0 & | & 1 \end{pmatrix}$$

The above system is inconsistent.

$$\begin{pmatrix} 1 & 3 & | & 1 \\ 0 & 1 & | & -1 \\ 0 & 0 & | & 0 \end{pmatrix}$$

The above system is consistent and has a unique solution

$$x = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 4 & | & 1 \\ 0 & 0 & 1 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

The above system is consistent, but has infinitely many solutions.

3. (1.2:3(a-c),) Find the solution.
52.24

$$\begin{pmatrix} 1 & 0 & 0 & | & -2 \\ 0 & 1 & 0 & | & 5 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \\ 0 & 0 & 0 & | & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 0 & | & 2 \\ 0 & 0 & 1 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

- **ans:**

$$\begin{pmatrix} 1 & 0 & 0 & | & -2 \\ 0 & 1 & 0 & | & 5 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}$$

The above system is consistent and has a unique solution

$$x = \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \\ 0 & 0 & 0 & | & 1 \end{pmatrix}$$

The above system is inconsistent.

$$\begin{pmatrix} 1 & -3 & 0 & | & 2 \\ 0 & 0 & 1 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

The above system is consistent, but has infinitely many solutions.

$$x = \begin{pmatrix} 3C+2 \\ C \\ -2 \end{pmatrix}$$

4. (1.2:5(b,e,i),) Find the reduced row echelon form and
52.26 solve the system.

$$\begin{aligned} 2x_1 - 3x_2 &= 5 \\ -4x_1 + 6x_2 &= 9 \end{aligned}$$

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ x_1 + x_2 + x_3 &= 3 \\ 3x_1 + 4x_2 + 2x_3 &= 4 \end{aligned}$$

$$\begin{aligned} -x_1 + 2x_2 - x_3 &= 2 \\ -2x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + 2x_3 &= 5 \\ -3x_1 + 8x_2 + 5x_3 &= 17 \end{aligned}$$

- **ans:**

$$\begin{aligned} 2x_1 - 3x_2 &= 5 \\ -4x_1 + 6x_2 &= 8 \end{aligned}$$

$$\begin{pmatrix} 2 & -3 & | & 5 \\ -4 & 6 & | & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 3/2 & | & 0 \\ 0 & 0 & | & 1 \end{pmatrix}$$

No solution.

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ x_1 + x_2 + x_3 &= 3 \\ 3x_1 + 4x_2 + 2x_3 &= 4 \end{aligned}$$

$$\begin{pmatrix} 2 & -3 & 1 & | & 1 \\ 1 & 1 & 1 & | & 3 \\ 3 & 4 & 2 & | & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & | & 8 \\ & 1 & -1 & | & 5 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Solution

$$x = \begin{pmatrix} 8-2C \\ -5+C \\ C \end{pmatrix}$$

$$\begin{aligned} -x_1 + 2x_2 - x_3 &= 2 \\ -2x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + 2x_3 &= 5 \\ -3x_1 + 8x_2 + 5x_3 &= 17 \end{aligned}$$

We can reduce the matrix to its RREF

$$\begin{pmatrix} -1 & 2 & -1 & | & 2 \\ -2 & 2 & 1 & | & 4 \\ 3 & 2 & 2 & | & 5 \\ -3 & 8 & 5 & | & 17 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ & 1 & 0 & | & 3/2 \\ & & 1 & | & 1 \\ & & 0 & | & 0 \end{pmatrix}$$

Unique solution

$$x = \begin{pmatrix} 0 \\ 3/2 \\ 1 \end{pmatrix}$$

5. (1.2:6(a),) Find the reduced row echelon form (Gauss-
52.28 Jordan reduction) and solve the system.

$$\begin{aligned} x_1 + x_2 &= -1 \\ 4x_1 - 3x_2 &= 3 \end{aligned}$$

- **ans:**

$$\begin{pmatrix} 1 & 1 & | & -1 \\ 4 & -3 & | & 3 \end{pmatrix} \overset{-4r_1+r_2}{\longrightarrow} \begin{pmatrix} 1 & 1 & | & -1 \\ & -7 & | & 7 \end{pmatrix}$$

$$\overset{(-1/7)r_2}{\longrightarrow} \begin{pmatrix} 1 & 1 & | & -1 \\ & 1 & | & -1 \end{pmatrix}$$

$$\overset{-r_2+r_1}{\longrightarrow} \begin{pmatrix} 1 & 0 & | & 0 \\ & 1 & | & -1 \end{pmatrix}$$

Solution

$$x = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

6. (1.2:8) For what values of a will the system have a
52.30 unique solution?

$$\begin{pmatrix} 1 & 2 & 1 & | & 1 \\ -1 & 4 & 3 & | & 2 \\ 2 & -2 & a & | & 3 \end{pmatrix}$$

- **ans:**

$$\begin{pmatrix} 1 & 2 & 1 & | & 1 \\ -1 & 4 & 3 & | & 2 \\ 2 & -2 & a & | & 3 \end{pmatrix} \overset{\substack{r_1+r_2 \\ -2r_1+r_3}}{\longrightarrow} \begin{pmatrix} 1 & 2 & 1 & | & 1 \\ & 6 & 4 & | & 3 \\ & -6 & a-2 & | & 1 \end{pmatrix}$$

$$\overset{r_2+r_3}{\longrightarrow} \begin{pmatrix} 1 & 2 & 1 & | & 1 \\ & 6 & 4 & | & 3 \\ & & a+2 & | & 4 \end{pmatrix}$$

When $a = -2$ there is no solution. Otherwise

$$x = \begin{pmatrix} \frac{4}{3(a+2)} \\ \frac{3a-10}{6(a+2)} \\ \frac{4}{a+2} \end{pmatrix}$$

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1. (1.3:1(c-f),) Find $2A - 3B$, $(aA)^T - (3B)^T$, AB, BA.
54.20

$$A = \begin{pmatrix} 3 & 1 & 4 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 1 & 1 \\ 2 & -4 & 1 \end{pmatrix}$$

- **ans:**

$$2A - 3B = \begin{pmatrix} 3 & 2 & 2 \\ 5 & -3 & -1 \\ -4 & 16 & 1 \end{pmatrix}$$

$$(aA)^T - (3B)^T = \begin{pmatrix} 3 & 5 & -4 \\ 2 & -3 & 16 \\ 2 & -1 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 8 & -15 & 11 \\ 0 & -4 & -3 \\ -1 & -6 & 6 \end{pmatrix}$$

$$BA = \begin{pmatrix} 5 & 5 & 8 \\ -10 & -1 & -9 \\ 15 & 4 & 6 \end{pmatrix}$$

2. (1.3:2(a,b),) Find AB
54.22

$$\begin{pmatrix} 3 & 5 & 1 \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 \\ 6 & -4 \\ 8 & -6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

- **ans:**

$$\begin{pmatrix} 3 & 5 & 1 \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 19 \\ 4 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -2 \\ 6 & -4 \\ 8 & -6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \text{not defined}$$

3. (1.3:4) Write the system as a matrix equation.
54.24

$$\begin{aligned} 3x_1 + 2x_2 &= 1 \\ 2x_1 - 3x_2 &= 5 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 \qquad &= 5 \\ 2x_1 + x_2 - x_3 &= 6 \\ 3x_1 - 2x_2 + x_3 &= 7 \end{aligned}$$

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4 \\ x_1 - x_2 + 2x_3 &= 2 \\ 3x_1 - 2x_2 - x_3 &= 0 \end{aligned}$$

- **ans:** $Ax = b$, where A, x, b are

$$\begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$Ax = b$, where A, x, b are

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 3 & -2 & 2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

$Ax = b$, where A, x, b are

$$\begin{pmatrix} 271 & 1 & \\ 1 & -1 & 2 \\ 3 & -2 & -1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

4. (1.3:11) Find A^2, A^3, A^{2n}, A^{2n+1}
54.26

$$A = \begin{pmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}$$

- **ans:**

$$\begin{aligned} A^2 &= I \\ A^3 &= A \\ A^{2n} &= I \\ A^{2n+1} &= A \end{aligned}$$

5. (1.3:12) Show $A^n = 0$ for $n \geq 4$,
54.28

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- **<u>ans:</u>**

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A^4 = 0$$

$$A^5 = A^4 A = 0A = 0$$

$$A^n = A^4 A^{n-4} = 0A^{n-4} = 0, \ n \geq 4$$

6. (1.3:14) Find the consistency by showing if $\mathbf{b}$ is a linear
54.30 combination of column vectors of A:

$$A = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

- **<u>ans:</u>** Find the solution of $Ax = b$, we get

$$\begin{pmatrix} 2 & 1 & | & 3 \\ -2 & -1 & | & -1 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & | & 3 \\ & 0 & | & 2 \end{pmatrix}$$

No solution. So not consistent.

Solve the system $Ax = b$, we get a unique solution.

$$\mathbf{b} = (1)\mathbf{a}_1 + (1)\mathbf{a}_2$$

Find the solution of $Ax = b$, we get

$$\begin{pmatrix} 3 & 2 & 1 & | & 1 \\ 3 & 2 & 1 & | & 0 \\ 3 & 2 & 1 & | & -1 \end{pmatrix} \to \begin{pmatrix} 3 & 2 & 1 & | & 1 \\ & & 0 & | & -1 \\ & & 0 & | & -2 \end{pmatrix}$$

No solution. So not consistent.

7. (1.3:20) Explain why each of the following algebraic
54.32 rules will not work in general when the real numbers are replaced by matrices.

$$(a) \ (a+b)^2 = a^2 + 2ab + b^2$$
$$(b) \ (a+b)(a-b) = a^2 - b^2$$

- **<u>ans:</u>** The idea is to find example a and b so that $ab \neq ba$.

Let

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \ b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$(a+b)^2 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$
$$a^2 + 2ab + b^2 = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$$

$$(a+b)(a-b) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$a^2 - b^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

8. (1.3:22) Find 2×2 matrices A and B that both are not
54.34 the zero matrix for which $AB = 0$.

- **<u>ans:</u>**

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Both nonzero. But $AB = 0$.