M341 H6 (S. Zhang) 4.1,1.1.

1. (4.1:1) Transform the differential equation into an
46.24 equivalent system of first-order differential equations.

$$x'' + 3x' + 7x = t^2$$

- **ans:**

$$\begin{aligned} x' &= y \\ y' &= -7x - 3y + t^2 \end{aligned}$$

2. (4.1:3) Transform the differential equation into an
46.26 equivalent system of first-order differential equations.

$$t^2x'' + tx' + (t^2 - 1)x = 0$$

- **ans:**

$$\begin{aligned} x' &= y \\ y' &= -\frac{t^2-1}{t^2}x - \frac{1}{t}y \end{aligned}$$

3. (4.1:6) Transform the differential equation(s) into an
46.28 equivalent system of first-order differential equations.

$$\begin{aligned} x'' - 5x + 4y &= 0, \\ y'' + 4x - 5y &= 0 \end{aligned}$$

- **ans:**

$$\begin{aligned} x' &= z \\ y' &= w \\ z' &= 5x - 4y \\ w' &= -4x + 5y \end{aligned}$$

4. (4.1:11) Solve the first order system by elimination
46.30

$$x' = y,\ y' = -x$$

- **ans:**

$$\begin{aligned} x'' = y' &= -x \\ x'' + x &= 0 \\ x &= c_1 \cos t + c_2 \sin t \\ y = x' &= -c_1 \sin t + c_2 \cos t \\ \begin{pmatrix} x \\ y \end{pmatrix} &= c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \end{aligned}$$

5. (4.1:12) Solve the first order system by elimination
46.32

$$x' = y,\ y' = x$$

- **ans:**

$$\begin{aligned} x'' = y' &= x \\ x'' - x &= 0 \\ x &= c_1e^t + c_2e^{-t} \\ y = x' &= c_1e^t - c_2e^{-t} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= c_1 \begin{pmatrix} e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} \end{aligned}$$

6. (4.1:19) Solve the first order system by elimination
46.34

$$\begin{aligned} x' &= -y \qquad x'(0) = 0 \\ y' &= 13x + 4y; y'(0) = 3 \end{aligned}$$

- **ans:**

$$\begin{aligned} x'' = -y' &= -13x - 4y = -13x + 4x' \\ x'' - 4x' + 13x &= 0 \end{aligned}$$

$$\begin{aligned} r &= 2 \pm 3i \\ x &= e^{2t}(c_1 \cos 3t + c_2 \sin 3t) \\ x(0) &= 0 \\ x &= e^{2t}c_2 \sin 3t \\ y = x' &= -e^{2t}(3c_2 \cos 3t + c_2 \sin 3t) \\ y(0) &= 3 \\ y = x' &= -e^{2t}(-3 \cos 3t - 2 \sin 3t) \end{aligned}$$

7. (4.1:20) Solve the first order system by elimination
46.36

$$x' = y,\ y' = -9x + 6y$$

- **ans:**

$$\begin{aligned} x'' = y' &= -9x + 6x' \\ x'' - 6x' + 9x &= 0 \\ r &= 3, 3 \\ x &= (A + Bt)e^{3t} \\ y = x' &= (3A + B + 3Bt)e^{3t} \end{aligned}$$

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1. (1.1:1c) Use back substitution to solve
50.20

$$\begin{array}{rrrrr} x_1+ & 3x_2+ & 3x_3+ & x_4 = & 5 \\ & 3x_2+ & x_3- & 3x_4 = & 1 \\ & - & x_3+ & 3x_4 = & -1 \\ & & & 4x_4 = & 4 \end{array}$$

- **ans:** From last equation, work upward.

$$\begin{aligned} x_4 &= 1 \\ x_3 &= 3 \\ x_2 &= 0 \\ x_1 &= -2 \end{aligned}$$

2. (1.1:2c) Find the coefficient matrix
50.22

$$\begin{array}{rrrrr} x_1+ & 3x_2+ & 3x_3+ & x_4 = & 5 \\ & 3x_2+ & x_3- & 3x_4 = & 1 \\ & - & x_3+ & 3x_4 = & -1 \\ & & & 4x_4 = & 4 \end{array}$$

- **ans:**

$$A = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 3 & 1 & -3 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

3. (1.1:3) Interpret each equation as a line and deter-
50.24 mine geometrically the number of solutions.

$$\begin{aligned} x_1 + x_2 &= 4 \\ x_1 - x_2 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ -2x_1 - 4x_2 &= 4 \end{aligned}$$

$$\begin{aligned} 2x_1 - x_2 &= 3 \\ -4x_1 + 2x_2 &= -6 \end{aligned}$$

- **ans:** (a) Two crossing lines. Intersection $(3, 1)$.

(b) Two parallel lines. No intersection/solution.

(c) Two overlapping lines. All points $C(1, -1)$ are solutions.

4. (1.1:6b) Solve the system by row operations.
50.25

$$\begin{array}{rr} 2x_1 & +x_2 = 8 \\ 4x_1 & -3x_2 = 6 \end{array}$$

- **ans:**

$$\begin{pmatrix} 2 & 1 & | & 8 \\ 4 & -3 & | & 6 \end{pmatrix}$$
$$\overset{-2r_1+r_2 \to r_2}{\longrightarrow} \begin{pmatrix} 2 & 1 & | & 8 \\ & -5 & | & -10 \end{pmatrix}$$

Solution

$$x = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

5. (1.1:6d) Solve the system by row operations.
50.26

$$\begin{array}{rrr} x_1 & +2x_2 & -x_3 = 1 \\ 2x_1 & -x_2 & +x_3 = 3 \\ -x_1+ & 2x_2+ & 3x_3 = 7 \end{array}$$

- **ans:**

$$\begin{pmatrix} 1 & 2 & -1 & | & 1 \\ 2 & -1 & 1 & | & 3 \\ -1 & 2 & 3 & | & 7 \end{pmatrix}$$
$$\overset{\substack{-2r_1+r_2 \\ r_1+r_3}}{\longrightarrow} \begin{pmatrix} 1 & 2 & -1 & | & 1 \\ & -5 & 3 & | & 1 \\ & 4 & 2 & | & 8 \end{pmatrix}$$
$$\overset{(4/5)r_2+r_3}{\longrightarrow} \begin{pmatrix} 1 & 2 & -1 & | & 1 \\ & -5 & 3 & | & 1 \\ & & 22/5 & | & 44/5 \end{pmatrix}$$

Solution

$$x = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

6. (1.1:7) Solve the two systems by doing row operations
50.27 together.

$$\begin{array}{rrcrr} 2x_1 & +x_2 = 3 & & 2x_1 & +x_2 = -1 \\ 4x_1 & +3x_2 = 5 & & 4x_1 & +3x_2 = 1 \end{array}$$

- **ans:**

$$\begin{pmatrix} 2 & 1 & | & 3 & -1 \\ 4 & 3 & | & 5 & 1 \end{pmatrix}$$
$$\overset{-2r_1+r_2 \to r_2}{\longrightarrow} \begin{pmatrix} 2 & 1 & | & 3 & -1 \\ & 1 & | & -1 & 3 \end{pmatrix}$$

Do back-substitution separately

$$x = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad x = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

7. (1.1:8) Solve the two systems by doing row operations
50.28 together.

$$\begin{array}{rrr} x_1 & +2x_2 & -2x_3 = 1 \\ 2x_1 & +5x_2 & +x_3 = 9 \\ x_1+ & 3x_2+ & 4x_3 = 9 \end{array}$$

$$\begin{array}{rrr} x_1 & +2x_2 & -2x_3 = 9 \\ 2x_1 & +5x_2 & +x_3 = 9 \\ x_1+ & 3x_2+ & 4x_3 = -2 \end{array}$$

- **ans:**

$$\begin{pmatrix}1 & 2 & -2 & | & 1 & 9\\ 2 & 5 & 1 & | & 9 & 9\\ 1 & 3 & 4 & | & 9 & -2\end{pmatrix}$$

$$\overset{-2r_1+r_2\to r_2}{\longrightarrow}\begin{pmatrix}1 & 2 & -2 & | & 1 & 9\\ & 1 & 5 & | & 7 & -9\\ 1 & 3 & 4 & | & 9 & -2\end{pmatrix}$$

$$\overset{-r_1+r_3\to r_3}{\longrightarrow}\begin{pmatrix}1 & 2 & -2 & | & 1 & 9\\ & 1 & 5 & | & 7 & -9\\ & 1 & 6 & | & 8 & -11\end{pmatrix}$$

$$\overset{-r_2+r_3\to r_3}{\longrightarrow}\begin{pmatrix}1 & 2 & -2 & | & 1 & 9\\ & 1 & 5 & | & 7 & -9\\ & & 1 & | & 1 & -2\end{pmatrix}$$

Do back-substitution separately

$$x = \begin{pmatrix}-1\\ 2\\ 1\end{pmatrix}, \quad x = \begin{pmatrix}3\\ 1\\ -2\end{pmatrix}$$