M341 H5 (S. Zhang) 3.5-6.

1. Find the general solution:

40.30

$$y'' - y' - 2y = 3x + 4$$

• **ans:** 3.5:2

$$r^2 - r - 2 = 0$$
$$r = -1, 2$$

$$y_H = c_1 e^{-t} + c_2 e^{2t}$$

Since $r = 0$ is not a root,

$$y_p = (A + Bx)$$

Plug it into the equation,

$$-2A - B = 4$$
$$-2B = 3$$

$$y_p = -\frac{5}{4} - \frac{3}{2}x$$

$$y = y_H + y_p = c_1 e^{-t} + c_2 e^{2t} - \frac{5}{4} - \frac{3}{2}x$$

2. Find the general solution:

40.32

$$y'' + y' + y = \sin^2 x$$

• **ans:** 3.5:5

Rewrite

$$y'' + y' + y = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$r^2 + r + 1 = 0$$
$$r = \frac{-1 \pm \sqrt{3}i}{2}$$

$$y_H = e^{-t/2}(c_1 \cos\frac{\sqrt{3}t}{2} + c_2 \sin\frac{\sqrt{3}t}{2})$$

Since $r = 0$ and $r = 2i$ are not roots,

$$y_p = (A) + (B\cos 2x + C\sin 2x)$$

Plug it into the equation,

$$A = \frac{1}{2}$$
$$-3B + 2C = -\frac{1}{2}$$
$$2B - 3C = 0$$

$$y_p = \frac{1}{2} + \frac{3}{26}\cos 2x - \frac{1}{13}\sin 2x$$

$$y = y_H + y_p =$$
$$= e^{-t/2}(c_1 \cos\frac{\sqrt{3}t}{2} + c_2 \sin\frac{\sqrt{3}t}{2})$$
$$+ \frac{1}{2} + \frac{3}{26}\cos 2x - \frac{1}{13}\sin 2x$$

3. (3.5:11) Find the general solution (resolve all con-
40.34 stants):

$$y''' + 4y' = 3x - 1$$

• **ans:** For $y_H = y_c$ (complementary solution, or solution for the homogeneous equation), we find the roots of characteristic eq

$$r^3 + 4r = 0, \ r = 0, \pm 2i$$

$$y_H = c_1 + c_2 \cos 2x + c_3 \sin 2x$$

Since $r = 0$ is a root, we need an extra x in y_p form,

$$y_p = x(A + Bx)$$

Plug it into the equation,

$$4A = -1$$
$$8B = 3$$

$$y_p = x(-\frac{1}{4} + \frac{8}{3}x)$$

$$y = y_H + y_p =$$
$$= c_1 + c_2 \cos 2x + c_3 \sin 2x + x(-\frac{1}{4} + \frac{8}{3}x)$$

4. (3.5:21) Find a y_P form, but do not find the coeffi-
40.35 cients.

$$y'' - 2y' + 2y = e^x \sin x$$

• **ans:** For $y_H = y_c$ (complementary solution, or solution for the homogeneous equation), we find the roots of characteristic eq

$$r^2 - 2r + 2 = 0, \ r = 1 \pm i$$

$$y_H = e^x(c_1 \cos x + c_2 \sin x)$$

Since $r = 1 + i$ is a root, we need an extra x in y_p form,

$$y_p = e^x(A\cos x + B\sin x)x$$

5. (3.5:22) Find a y_P form, but do not find the coeffi-
40.36 cients.

$$y^{(5)} - y^{(3)} = e^x + 2x^2 - 5$$

• **ans:** For $y_H = y_c$ (complementary solution, or solution for the homogeneous equation), we find the roots of characteristic eq

$$r^5 - r^3 = 0,\ r = 0, 0, 0, 1, -1$$

Since $r = 1$ is a root, we need an extra x in y_p form for the e^x term, and because 0 is a triple root, we need an extra x^3 in y_p form for the $(2x^2 - 5)$ term, ,

$$y_p = x(Ae^x) + x^3(B + Cx + Dx^2)$$

6. (3.5:23) Find a y_P form, but do not find the coeffi-
40.55 cients.

$$y'' + 4y = 3x \cos 2x$$

• **ans:** For $y_H = y_c$ (complementary solution, or solution for the homogeneous equation), we find the roots of characteristic eq

$$r = \pm 2i$$

Since $r = 3 + 2i$ is not a root,

$$y_p = e^{3x}(A \cos 2t + B \sin 2t)$$

7. (3.5:24) Find a y_P form, but do not find the coeffi-
40.56 cients.

$$y''' - y'' - 12y' = x - 2xe^{-3x}$$

• **ans:** For $y_H = y_c$ (complementary solution, or solution for the homogeneous equation), we find the roots of characteristic eq

$$r = 0, 4, -3$$

Since $r = -3$ is a root,

$$y_p = (Ax + B) + e^{-3x}(CX + D)x$$

8. (3.5:25) Find a y_P form, but do not find the coeffi-
40.57 cients.

$$y'' + 3y' + 2y = x(e^{-x} - e^{-2x})$$

• **ans:** For $y_H = y_c$ (complementary solution, or solution for the homogeneous equation), we find the roots of characteristic eq

$$r = -1, -2$$

Since $r = -1, -2$ are both a root,

$$y_p = (Ax + B)e^{-x}x + e^{-2x}(CX + D)x$$

9. (3.5:26) Find a y_P form, but do not find the coeffi-
40.58 cients.

$$y'' - 6y' + 13y = x(e^{3x} \sin 2x)$$

• **ans:** For $y_H = y_c$ (complementary solution, or solution for the homogeneous equation), we find the roots of characteristic eq

$$r = 3 \pm 2i$$

Since $r = 3 + 2i$ is a root,

$$y_p = e^{3x}((Ax + B) \cos 2x + (Cx + D) \sin 2x)x$$

10. (3.5:27) Find a y_P form, but do not find the coeffi-
40.59 cients.

$$y^{(4)} + 5y'' + 4y = \sin x + \cos 2x$$

• **ans:** For $y_H = y_c$ (complementary solution, or solution for the homogeneous equation), we find the roots of characteristic eq

$$r^2 = -1, -4$$
$$r = \pm i, \pm 2i$$

Since both r are a root,

$$y_p = (A \cos x + B \sin x)x + (C \cos 2x + D \sin 2x)x$$

11. (3.5:28) Find a y_P form, but do not find the coeffi-
40.61 cients.

$$y^{(4)} + 9y'' = (x^2 + 1) \sin 3x$$

• **ans:** For $y_H = y_c$ (complementary solution, or solution for the homogeneous equation), we find the roots of characteristic eq

$$r = 0, 0, \pm 3i$$

Since both r are a root,

$$y_p = [(Ax^2 + Bx + C) \cos 3x + (Dx^2 + Ex + F) \sin 3x]x$$

12. (3.5:29) Find a y_P form, but do not find the coeffi-
40.62 cients.

$$(D - 1)^3(D^2 - 4)y = xe^x + e^{2x} + e^{-2x}$$

• **ans:** For $y_H = y_c$ (complementary solution, or solution for the homogeneous equation), we find the roots of characteristic eq

$$r = 1, 1, 1, 2, -2$$

Since both $r = 1$ a 3-fold root,

$$y_p = (Ax + B)e^x x^2 + Ce^{2x}x$$

13. (3.5:32) Solve IVP

40.38
$$y'' + 3y' + 2y = e^x;\ y(0) = 0,\ y'(0) = 3$$

• **ans:** For $y_H = y_c$ (complementary solution, or solution for the homogeneous equation), we find the roots of characteristic eq

$$r^2 + 3r + 2 = 0,\ r = -1, -2$$

$$y_H = c_1 e^{-x} + c_2 e^{-2x}$$

Since $r = 1$ is a root, we have no extra x in y_p form,

$$y_p = Ae^x$$

Plug it into the equation,

$$A = \frac{1}{6}$$

$$y_p = \frac{1}{6} e^x$$

$$\begin{aligned} y &= y_H + y_p = \\ &= c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{6} e^x \end{aligned}$$

By the initial conditions

$$\begin{aligned} c_1 + c_2 + \frac{1}{6} &= 0 \\ -c_1 - 2c_2 + \frac{1}{6} &= 3 \end{aligned}$$

$$y = \frac{5}{2} e^{-x} - \frac{8}{3} e^{-2x} + \frac{1}{6} e^x$$

14. (3.5:34) Solve IVP

40.40
$$y'' + y = \cos x;\ y(0) = 1,\ y'(0) = -1$$

• **ans:** For $y_H = y_c$ (complementary solution, or solution for the homogeneous equation), we find the roots of characteristic eq

$$r^2 + 1 = 0,\ r = \pm i$$

$$y_H = c_1 \cos x + c_2 \sin x$$

Since $r = i$ is a root, we have no extra x in y_p form,

$$y_p = x(A \cos x + B \sin x)$$

Plug it into the equation,

$$y_p = \frac{1}{2} x \sin x$$

$$\begin{aligned} y &= y_H + y_p = \\ &= c_1 \cos x + c_2 \sin x + \frac{1}{2} x \sin x \end{aligned}$$

By the initial conditions

$$y = \cos x - \sin x + \frac{1}{2} x \sin x$$

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1. (3.6:1) Write the solution as sum to two motions.

42.30 Find its period.

$$x'' + 9x = 10 \cos 2t,\ x(0) = x'(0) = 0$$

• **ans:** For $x_c = x_H$,

$$r^2 + 9 = 0;\ r = \pm 3i$$

$$x_H = c_1 \cos 3t + c_2 \sin 3t$$

Since $r = 2i$ is not a root, we set up an x_P form

$$x_P = A \cos 2t + B \sin 2t$$

Plug it into the equation ,we get

$$x_P = 2 \cos 2t$$

$$\begin{aligned} x =& x_H + x_P \\ =& c_1 \cos 3t + c_2 \sin 3t + 2 \cos 2t \end{aligned}$$

By the initial conditions,

$$x = -2 \cos 3t + 2 \cos 2t$$

There are two oscillations. The periods are $\frac{2\pi}{3}$ and $\frac{2\pi}{2}$, respectively. The period of the motion is the least common multiple of the two periods, which is 2π.

2. (3.6:11) For the forced oscillation

42.32
$$mx'' + cx' + kx = F(t) = F_0 \cos \omega t$$

suppose

$$m = 1,\ c = 4,\ k = 5,\ F_0 = 10,\ \omega = 3$$

$$x(0) = 0,\ x'(0) = 0$$

Find the transient motion (i.e. $x_c = x_H$ part) and the steady periodic oscillation (i.e. $x_P = x_{sp}$).

• **ans:**

$$x'' + 4x' + 5x = 10 \cos 3t$$

For $x_c = x_H$,

$$r^2 + 4r + 5 = 0;\ r = -2 \pm i$$

$$x_H = e^{-2t}(c_1 \cos t + c_2 \sin t)$$

Since $r = 3i$ is not a root, we set up an x_P form

$$x_P = A\cos 3t + B \sin 3t$$

$$\begin{aligned} -4A + 12B &= 10 \\ 12A + 4B &= 0 \end{aligned}$$

$$\begin{aligned} x_P =& x_{sp} = -\frac{1}{4}\cos 3t + \frac{3}{4}\sin 3t \\ =& \frac{\sqrt{10}}{4}\cos(3t - \delta) \\ \delta =& \pi - \tan^{-1} 3 \simeq 1.8925 \end{aligned}$$

$$\begin{aligned} x =& x_H + x_P \\ =& e^{-2t}(c_1 \cos t + c_2 \sin t) + x_P \end{aligned}$$

By the initial conditions,

$$\begin{aligned} c_1 - \frac{1}{4} &= 0 \\ -2c_1 + c_2 + \frac{9}{4} &= 0 \end{aligned}$$

We find

$$c_1 = \frac{1}{4}, \; c_2 = -\frac{7}{4}$$

The transient motion (disappear after a while)

$$\begin{aligned} x_{tr} =& x - x_P = e^{-2t}(\frac{1}{4}\cos t - \frac{7}{4}\sin t) \\ =& \frac{5}{4}\sqrt{2}e^{-2t}\cos(t - \beta) \\ \beta =& 2\pi - \tan^{-1} 7 = 4.8543 \end{aligned}$$

3. (3.6:16) For the forced oscillation

42.34

$$mx'' + cx' + kx = F(t) = F_0 \cos \omega t$$

suppose

$$m = 1, \; c = 4, \; k = 5, \; F_0 = 10,$$

Find the practical resonance frequency ω for the given m, c, k, and F_0.

• **ans:** Since $r = \omega i$ is not a root of the characteristic equation (because $c \neq 0$), we set

$$x_P = x_{sp} = A\cos \omega t + B \sin \omega t$$

Plug it into the equation

$$\begin{aligned} (k - m\omega^2)A + c\omega B &= F_0 \\ -c\omega A + (k - m\omega^2)B &= 0 \end{aligned}$$

We get

$$\begin{aligned} A &= \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2} \\ B &= \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2} \end{aligned}$$

$$\begin{aligned} x_P = x_{sp} &= A\cos \omega t + B \sin \omega t \\ &= C\cos(\omega t - \delta) \\ C(\omega) &= \sqrt{A^2 + B^2} \\ &= \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \end{aligned}$$

When c is small and $k \sim m\omega^2$ (i.e., $\omega \sim \omega_0$), the amplitude is much bigger.

Plugging in the given data, we have

$$\begin{aligned} C(\omega) &= \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \\ &= \frac{10}{\sqrt{25 + 6\omega^2 + \omega^4}} \\ C'(\omega) &= \frac{6\omega + 2\omega^3}{(25 + 6\omega^2 + \omega^4)^{3/2}} \neq 0 \end{aligned}$$

The graph of $C(\omega)$ starts from $C(0) = 2$ and decreases all the way down. So there is no practical resonance frequency.

4. (3.6:17) For the forced oscillation

42.36

$$mx'' + cx' + kx = F(t) = F_0 \cos \omega t$$

suppose

$$m = 1, \; c = 6, \; k = 45, \; F_0 = 50,$$

Find the practical resonance frequency ω for the given m, c, k, and F_0.

• **ans:** Since $r = \omega i$ is not a root of the characteristic equation (because $c \neq 0$), we set

$$x_P = x_{Sp} = A\cos \omega t + B \sin \omega t$$

Plug it into the equation

$$\begin{aligned} (k - m\omega^2)A + c\omega B &= F_0 \\ -c\omega A + (k - m\omega^2)B &= 0 \end{aligned}$$

We get

$$\begin{aligned} A &= \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2} \\ B &= \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2} \end{aligned}$$

$$
\begin{aligned}
x_P = x_{sp} &= A\cos\omega t + B\sin\omega t \\
&= C\cos(\omega t - \delta) \\
C(\omega) &= \sqrt{A^2 + B^2} \\
&= \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}
\end{aligned}
$$

When c is small and $k \sim m\omega^2$ (i.e., $\omega \sim \omega_0$), the amplitude is much bigger.

Plugging in the given data, we have

$$
\begin{aligned}
C(\omega) &= \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \\
&= \frac{50}{\sqrt{2025 - 54\omega^2 + \omega^4}} \\
C'(\omega) &= \frac{-100\omega(-27 + \omega^2)}{(2025 - 54\omega^2 + \omega^4)^{3/2}} \\
&= 0 \\
\omega &= \sqrt{27}
\end{aligned}
$$

So the practical resonance frequency is $\omega = \sqrt{27}$.

5. (3.6:20) A front-loading washing machine is mounted
42.38 on a thick rubber pad that acts like a spring; the weight $W = mg$ (with $g = 9.8m/s^2$) of the machine depresses the pad exactly $0.5cm$. When its rotor spins at ω radians per second, the rotor exerts a vertical force $F_0 \cos\omega t$ newtons on the machine. At what speed (in revolutions per minute) will resonance vibrations occur? Neglect friction.

- **ans:** By Hooke's law

$$F = kx,\ \ mg = k(1/200),\ \ k = 200mg(N/m)$$

The resonance frequency is the same as the natural frequency

$$
\begin{aligned}
\omega = \omega_0 &= \sqrt{\frac{k}{m}} \\
&= \sqrt{200g} \simeq 44.27\frac{rad}{sec} \simeq 7.05Hz
\end{aligned}
$$

That is one round per 7.05 sec. So the resonance rotation speed is $7.05 \times 60 = 423$ rounds per minute.

6. (3.6:23) A building consists of two floors. The first
42.40 floor is attached rigidly to the ground and the second floor is of mass $m = 1000$ slugs (fps units) and weighs 16 tons (32,000 lb). The elastic frame of the building behaves as a spring that resists horizontal displacements of the second floor; it requires a horizontal force of 5 tons to displace the second floor a distance of 1ft. Assume that in an earthquake the ground oscillates horizontally with amplitude A_0 and circular frequency ω, resulting in an external horizontal force

$$F(t) = mA_0\omega^2 \sin\omega t$$

on the second floor.

(a) What is the natural frequency in hertz of oscillations of the second floor?

(b) If the ground undergoes one oscillation every 2.25s with an amplitude of 3in., what is the amplitude of the resulting forced oscillations of the second floor?

- **ans:** (a) In ft-lb-sec units we have

$$
\begin{aligned}
m = 1000&,\ \ k = 10000, \\
\omega_0 = \sqrt{\frac{k}{m}} &= \sqrt{10}\frac{rad}{sec} \simeq 0.5Hz
\end{aligned}
$$

(b)

$$\omega = \frac{2\pi}{2.25} \simeq 2.79$$

The DE

$$
\begin{aligned}
mx'' + kx &= mA_0\omega^2 \sin\omega t \\
1000x'' + 10000x &= 1000\frac{3}{12}\omega^2 \sin\omega t \\
x'' + 10x &= \frac{1}{4}\omega^2 \sin\omega t \\
x_P &= A\sin\omega t + B\cos\omega t \\
x_P &= \frac{\omega^2}{4(10 - \omega^2)}\sin\omega t \\
\frac{\omega^2}{4(10 - \omega^2)} &\simeq 0.8854ft = 10.63in
\end{aligned}
$$

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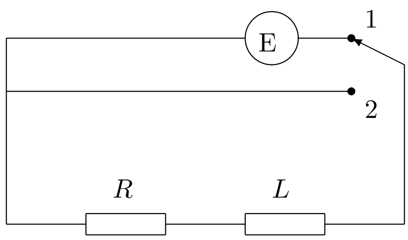
1. (§3.7:2) Suppose that the switch is initially in position
44.20 2, but is thrown to position 1 at time $t = 0$, so that $I(0) = 0$ and $E = 100$ for $t > 0$. Find $I(t)$ and show that $I(t) \to 4$ when $t \to \infty$.

$$L = 5H, \quad R = 25\Omega.$$



- **ans:**

$$5I' + 25I = 100$$

$$I_H = Ce^{-5t}$$

$$I_P = A = 4$$

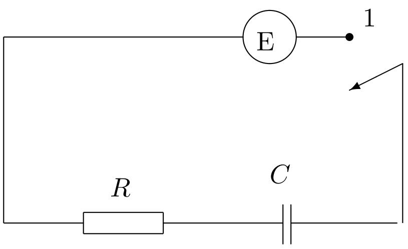
By initial condition,

$$I = I_H + I_P = Ce^{-5t} + 4 = -4e^{-5t} + 4 \to 4.$$

We can solve it as a first order linear equation.

2. (§3.7:8) Suppose that $Q(0) = 0$. Find $Q(t)$ and $I(t)$.
44.22 What is the maximum charge on the capacitor for $t \geq 0$ and when does it occur?

$$E(t) = 100e^{-5t}, \quad R = 10, \quad C = 0.02.$$



- **ans:**

$$Q' + 5Q = 10e^{-5t}$$

$$Q_H = Ce^{-5t}$$

$$Q_P = tAe^{-5t} = t10e^{-5t}$$

By initial condition,

$$Q = Ce^{-5t} + t10e^{-5t} = t10e^{-5t}$$

We can solve it as a first order linear equation.

$$\rho = e^{5t}$$

$$I(t) = Q'(t) = 10(1 - 5t)e^{-5t}$$

$Q_{\max}$ occurs at first $I(t) = 0$.

$$I(t) = Q'(t) = 10(1 - 5t)e^{-5t} = 0, \; t = \frac{1}{5}$$

$$Q_{\max} = Q(\frac{1}{5}) = \frac{2}{e}$$

3. (EP§3.713) In an RLC circuit with input voltage
44.24 $E(t)$, find $I_sp(t)$ in the standard form.

$$R = 20\Omega, \; L = 5H, \; C = 0.01F; \; E(t) = 200\cos 5tV.$$

- **ans:**

$$LI'' + RI' + C^{-1}I = E'(t)$$

$$5I'' + 20I' + 100I = -1000\sin 5t$$

$$I_P = I_{sp} = A\cos 5t + B\sin 5t$$

$$3A - 2B = 0, \; 2A + 3B = 20$$

$$\begin{aligned} I_P = I_{sp} &= \frac{40}{13}\cos 5t + \frac{60}{13}\sin 5t \\ &= \frac{20}{\sqrt{13}}(\frac{2}{\sqrt{13}}\cos 5t + \frac{3}{\sqrt{13}}\sin 5t) \\ &= \frac{20}{\sqrt{13}}\sin(5t - \delta) \\ \delta &= 2\pi - \tan^{-1}\frac{2}{3} \simeq 5.695 \end{aligned}$$