M341 H4 (S. Zhang) .

1. (3.3:1) Solve

36.1

$$y'' - 4y = 0$$

- **ans:** Characteristic equation

$$r^2 - 4 = 0,\ r = \pm 2$$

The general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x}$$

2. (3.3:2) Solve

36.2

$$2y'' - 3y' = 0$$

- **ans:** Characteristic equation

$$2r^2 - 3r = 0,\ r = 0, \frac{3}{2}$$

The general solution is

$$y = c_1 + c_2 e^{3x/2}$$

3. (3.3:10,) Solve

36.5

$$5y^{(4)} + 3y^{(3)} = 0$$

- **ans:** Characteristic equation

$$5r^4 + 3r^3 = 0,\ r = 0, 0, 0, -\frac{3}{5}$$

The general solution is

$$y = c_1 + c_2 x + c_3 x^2 + c_4 e^{-3x/5}$$

4. (3.3:14) Solve

36.6

$$y^{(4)} + 3y'' - 4y = 0$$

- **ans:** Characteristic equation

$$r^4 + 3r^2 - 4 = 0,\ r^2 = 1, -4,\ r = \pm 1, \pm 2i$$

The general solution is

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos 2x + c_4 \sin 2x$$

5. (3.3:21) Solve IVP

36.9

$$y'' - 4y' + 3y = 0,\ y(0) = 7,\ y'(0) = 11$$

- **ans:** Characteristic equation

$$r^2 - 4r + 3 = 0,\ r = 1, 3$$

The general solution is

$$y = c_1 e^{x} + c_2 e^{3x}$$

$$c_1 + c_2 = 7,\ c_1 + 3c_2 = 11$$

$$y = 5e^{x} + 2e^{3x}$$

6. (3.3:23) Solve IVP

36.10

$$y'' - 6y' + 25y = 0,\ y(0) = 3,\ y'(0) = 1$$

- **ans:** Characteristic equation

$$r^2 - 6r + 25 = 0,\ r = 3 \pm 4i$$

The general solution is

$$y = c_1 e^{3x} \cos 4x + c_2 e^{3x} \sin 4x$$

$$c_1 = 3,\ 3c_1 + 4c_2 = 1$$

$$y = 3e^{3x} \cos 4x - 2e^{3x} \sin 4x$$

7. (3.3:24) Solve IVP

36.11

$$\begin{aligned} 2y''' - 3y'' - 2y' = 0, \\ y(0) = 1,\ y'(0) = -1,\ y''(0) = 3 \end{aligned}$$

- **ans:** Characteristic equation

$$2r^3 - 3r^2 - 2y = 0,\ r = 0, 2, -1/2$$

The general solution is

$$y = c_1 + c_2 e^{2x} + c_3 e^{-x/2}$$

$$\begin{aligned} c_1 + c_2 + c_3 = 1,\ 2c_2 - c_3/2 = -1, \\ 4c_2 + c_3/4 = 3 \end{aligned}$$

$$y = -\frac{7}{2} + \frac{1}{2} e^{2x} + 4e^{-x/2}$$

8. (3.3:33) Given one solution, find the general solution.

36.13

$$y''' + 3y'' - 54y = 0,\ y = e^{3x}$$

- **ans:** Characteristic equation

$$r^3 + 3r^2 - 54 = 0$$

One root is $r = 3$,

$$r^3 + 3r^2 - 54 = (r - 3)(r^2 + 6r + 18)$$

$$r = 3, -3 \pm 3$$

The general solution is

$$y = c_1 e^{3x} + c_2 e^{3x} \cos 3x + c_3 e^{3x} \sin 3x$$

9. (3.3:35) Given one solution, find the general solution.

36.14 $$6y^{(4)} + 5y''' + 25y'' + 20y' + 4y = 0,\ y = \cos 2x$$

- **ans:** Characteristic equation

$$6r^4 + 5r^3 + 25r^2 + 20r + 4 = 0$$

Two root2 are $r = \pm 2i$,

$$6r^4 + 5r^3 + 25r^2 + 20r + 4 = (r^2 + 4)(6r^2 + 5t + 1)$$

$$r = \pm 2i, -1/2, -1/3$$

The general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + c_3 e^{-x/2} + c_4 e^{-x/3}$$

10. (3.3:39) Find a linear homogeneous constant-
36.17 coefficient equation with the given solution.

$$y(x) = (A + Bx + Cx^2)e^{2x}$$

- **ans:** Roots are

$$r = 2, 2, 2$$

$$(r - 2)^3 = r^3 - 6r^2 + 12r - 8$$

DE:

$$y''' - 6y'' + 12y' - 8y = 0$$

11. (3.3:40) Find a linear homogeneous constant-
36.19 coefficient equation with the given solution.

$$y(x) = Ae^{2x} + B\cos 2x + C \sin 2x$$

- **ans:** Roots are

$$r = 2, \pm 2i$$

$$(r - 2)(r^2 + 4) = 0$$

DE:

$$(D - 2)(D^2 + 4)y = 0$$

12. (3.3:42) Find a linear homogeneous constant-
36.21 coefficient equation with the given solution.

$$y(x) = (A + Bx + Cx^2)\cos 2x + (D + Ex + Fx^2)\sin 2x$$

- **ans:** Roots are

$$r = \pm 2i, \pm 2i, \pm 2i,$$

$$(r^2 + 2^2)^3 = 0$$

DE:

$$(D^2 + 4)^3 y = 0$$

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1. Determine the period and frequency of the simple har-
38.1 monic motion of a 4-kg mass on the end of a pring with spring constant 16N/m.

- **ans:** EP §3.4: 1

Freqency

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{48}{0.75}} = 8\frac{\text{rad}}{\text{sec}} = \frac{4}{\pi}\text{Hz}$$

Period

$$T = \frac{2\pi}{\omega_0} = \frac{\pi}{4}\text{sec}$$

2. A mass of 3kg is attached the end of a pring that is
38.5 stretched 20 cm by a force of 15 N. It is set in motion with initial position $x_0 = 0$ and initial velocity $v_0 = -10$m/s. Find the amplitude, period, and frequency of the resulting motion.

- **ans:** EP §3.4: 3

The spring constant

$$k = 15/0.2 = 75N/m$$

The solution for

$$3x'' + 75x = 0,\ x(0) = 0,\ x'(0) = -10$$

is

$$x(t) = -2\sin 5t$$

The amlitude is 2 m; the frequency is

$$\omega_0 = \sqrt{\frac{k}{m}} = 5rad/sec = \frac{2.5}{\pi}Hz$$

The period is

$$T = \frac{2\pi}{5}sec$$

3. Assume that the equation for a simple pendulum of length
38.9 L is

$$L\theta'' + g\theta = 0$$

where $g = GM/R^2$. Two pendulums are of lengths L_1 and L_2 and – when located at the respetive distances R_1 and R_2 from the center of the earth – having periods p_1 and p_2. Show that

$$\frac{p_1}{p_2} = \frac{R_1\sqrt{L_1}}{R_2\sqrt{L_2}}$$

- **ans:** EP §3.4: 5

When θ small, the equation for a pendulum is

$$\theta'' + \frac{g}{L}\theta = 0$$

$$g = GM/R^2$$

The circular frequency ω of a pendulum is

$$\omega^2 = \frac{g}{L} = \frac{GM}{R^2 L}$$

So the period is

$$p = \frac{2\pi}{\omega} = 2\pi R\sqrt{\frac{L}{GM}}$$

Now

$$\begin{aligned}\frac{p_1}{p_2} &= \frac{2\pi R_1\sqrt{\frac{L_1}{GM}}}{2\pi R_2\sqrt{\frac{L_2}{GM}}} \\ &= \frac{R_1\sqrt{L_1}}{R_2\sqrt{L_2}}\end{aligned}$$

4. For a mass attached to a spring and a dashpot, find $x(t)$.
38.17 Determine whetcher the motion is overdamped, critically damped, or underdamped. If it is underdamped, write the position function in the form

$$x(t) = C_1 e^{-pt}\cos(\omega_1 t - \alpha_1)$$

Also, find the undamped position function

$$u(x) = C_0\cos(\omega_0 t - \alpha_0)$$

that would result if the mass on the spring were set in motion with the same initial position and velocity, but with the dashpot disconnected. Finally construct a figure that illustrates the effect of damping by comparing the graphs of $x(t)$ and $u(t)$.

$$m = \frac{1}{2},\ c = 3,\ k = 4,\ x_0 = 2,\ v_0 = 0$$

- **ans:** EP §3.4: 15

$$\begin{aligned}\frac{1}{2}x'' + 3x' + 4x &= 0 \\ r &= -2,\ -4 \\ x(t) &= Ae^{-2t} + Be^{-4t} \\ &= 4e^{-2t} - 2e^{-4t}\end{aligned}$$

Overdamped.

If no damping,

$$\begin{aligned}\frac{1}{2}x'' + 4x &= 0 \\ r &= \pm 2\sqrt{2}i \\ x(t) &= A\cos(2\sqrt{2}t) + B\sin(2\sqrt{2}t) \\ &= 2\cos(2\sqrt{2}t - 0)\end{aligned}$$

5. For a mass attached to a spring and a dashpot, find $x(t)$.
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that would result if the mass on the spring were set in motion with the same initial position and velocity, but with the dashpot disconnected. Finally construct a figure that illustrates the effect of damping by comparing the graphs of $x(t)$ and $u(t)$.

$$m = 2,\ c = 12,\ k = 50,\ x_0 = 0,\ v_0 = -8$$

- **ans:** EP §3.4: 18

$$\begin{aligned}2x'' + 12x' + 50x &= 0 \\ r &= -3 \pm 4i \\ x(t) &= Ae^{-3t}\cos 4t + Be^{-3t}\sin 4t \\ &= -2e^{-3t}\sin 4t \\ &= 2e^{-3t}\cos(4t - \frac{3\pi}{2})\end{aligned}$$

Underdamped.

If no damping,

$$\begin{aligned}2x'' + 50x &= 0 \\ r &= \pm 5i \\ x(t) &= A\cos(5t) + B\sin(5t) \\ &= -\frac{8}{5}\sin 5t = \frac{8}{5}\cos(5t - \frac{3\pi}{2})\end{aligned}$$

6. (Underdamped) Show that the local maximum and min-
38.23 ima of

$$x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha)$$

occur where

$$\tan(\omega_1 t - \alpha) = -\frac{p}{\omega_1}$$

Conclude that

$$t_2 - t_1 = \frac{2\pi}{\omega_1}$$

if two consecutive maxima occur at time t_1 and t_2.

- **ans:** EP §3.4: 32

For underdamped oscillation:

$$x = Ce^{-pt}\cos(\omega_1 t - \alpha)$$

we find its peak points where the derivative is zero:

$$\begin{aligned} x' &= -pCe^{-pt}\cos(\omega_1 t - \alpha) \\ &\quad - C\omega_1 e^{-pt}\sin(\omega_1 t - \alpha) \\ &= 0 \\ \tan(\omega_1 t - \alpha) &= -\frac{p}{\omega_1} \end{aligned}$$

The gap between two peaks is when

$$\begin{aligned} \tan(\omega_1 t - \alpha) &= \tan(\omega_2 t - \alpha) \\ \omega_1 t_2 &= \omega_2 t_1 + 2\pi \\ t_2 - t_1 &= \frac{2\pi}{\omega_1} \end{aligned}$$

7. (Underdamped) Let x_1 and x_2 be two consecutive local
38.25 maximum values of $x(t)$. Show

$$\ln\frac{x_1}{x_2} = \frac{2\pi p}{\omega_1}$$

The constant $\frac{2\pi p}{\omega_1}$ is called the logarithmic decrement of the oscillation.

- **ans:** EP §3.4: 33 For underdamped oscillation:

$$x = Ce^{-pt}\cos(\omega_1 t - \alpha)$$

we find its peak points where the derivative is zero:

$$\begin{aligned} x' &= -pCe^{-pt}\cos(\omega_1 t - \alpha) \\ &\quad - C\omega_1 e^{-pt}\sin(\omega_1 t - \alpha) \\ &= 0 \\ \tan(\omega_1 t - \alpha) &= -\frac{p}{\omega_1} \end{aligned}$$

The gap between two peaks is when

$$\begin{aligned} \tan(\omega_1 t - \alpha) &= \tan(\omega_2 t - \alpha) \\ \omega_1 t_2 &= \omega_2 t_1 + 2\pi \\ t_2 - t_1 &= \frac{2\pi}{\omega_1} \end{aligned}$$

At the two t_i, we have

$$\begin{aligned} x_1 &= Ce^{-pt_1}\cos(\omega_1 t_1 - \alpha) \\ x_2 &= Ce^{-pt_2}\cos(\omega_1 t_2 - \alpha) \\ \frac{x_1}{x_2} &= e^{p(t_2 - t_1)} \\ \ln\frac{x_1}{x_2} &= p(t_2 - t_1) = p\frac{2\pi}{\omega_1} \end{aligned}$$