M341 H3 (S. Zhang) [b].

1. (2.1:10) Suppose that when a certain lake is stocked
22.2 with fish, the birth and death rates β and δ are both proportional to $1/\sqrt{P}$. If $P_0 = 900$ and after 6 months, there are 441 fish in the lake, how long did it take all the fish in the lake to die?

   - **ans:** Let $k = (\beta - \delta)/\sqrt{P}$.

$$\frac{d}{dt}P = k\frac{1}{\sqrt{P}}P$$

$$\int \frac{1}{\sqrt{P}}dP = kdt$$

$$P(t) = \left(\frac{1}{2}kt + C\right)^2$$

$$P(t) = \left(\frac{1}{2}kt + \sqrt{P_0}\right)^2$$

$P(0) = 900$,P(6)=441,

$$441 = \left(\frac{1}{2}kt + 30\right)^2$$

then $k = -3$, and

$$P(t) = (-1.5t + 30)^2$$

$P = 0$, $t = 20$.

2. (2.1:12) The time rate of change of an alligator pop-
22.5 ulation P in a swamp is proportional to the square of P. The swamp contained a dozen alligators in 1988, two dozen in 1998. When will there be four dozen alligators in the swamp? What happens there after?

   - **ans:**

$$P' = kP^2, \quad -\frac{1}{P} = kt - C$$
$$P = \frac{1}{C - kt}$$
$$P(0) = 12,\ C = \frac{1}{12},\ P = \frac{12}{1 - 12kt}$$
$$P(10) = 24,\ k = \frac{1}{240},\ P = \frac{240}{20 - t}$$
$$P = 48, \quad t = 15$$
$$t \to 20, \quad P \to \infty$$

3. (2.1:21) Suppose that a logistic population (in mil-
22.6 lions) satisfies

$$P' = kP(200 - P)$$

The population in 1940 was 100 million and was then growing at the rate of 1 million per year. Predict the population for the year 2000.

   - **ans:** $M = 200$ The birth+death rate is proportional to the gap to the maximal population.

$$P' = kP(M - P)$$

$P'(0) = 1$ and $P(0) = 100$

$$1 = k100(200 - 100), \quad k = 0.0001$$

$$P' = kP(M - P) \Rightarrow P = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

$$P = \frac{200}{1 + e^{-t/50}}$$

$$P(60) = \frac{200}{1 + e^{-6/5}} = 153.7$$

4. (2.1:28) Suppose that the number $x(t)$, with t in
22.11 months, of alligators in a swamp satisfies the differential equation

$$x' = 0.0001x^2 - 0.01x$$

(a) If initially there are 25 alligators in the swamp, solve this differential equation to determine what happens to the alligator population in the long run.

Repeat part (a), except with 150 alligators initially.

   - **ans:**

$$x' = 0.0001x^2 - 0.01x = 0.0001x(x - 100)$$

$$\int \frac{dx}{x(x-100)} = \int 0.0001dt$$
$$\int \frac{dx}{x-100} - \frac{dx}{x} = \int 0.01dt$$
$$\ln\frac{|x-100|}{|x|} = \frac{1}{100}t + \ln C$$
$$\frac{|x-100|}{|x|} = Ce^{t/100}$$

$x(0) = 25$, $C = 3$,

$$x = \frac{100}{1 + 3e^{t/100}} \to 0$$

If $x(0) = 150$, $C = 1/3$,

$$x = \frac{300}{3 - e^{t/100}}$$

When $x \to \infty$ $t = 100\ln 3 = 109.86$.

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1. (2.3:2) Suppose that a body moves through a resisting
24.3 medium with resistance proportional to its velocity v, so that $v' = -kv$.

(a) Show that its velocity and position at t are given by

$$v(t) = v_0 e^{-kt}$$

$$x(t) = x_0 + \frac{v_0}{k}(1 - e^{-kt})$$

(b) Conclude that the body travels only a finite distance and find that distance.

(c) Conclude that the body travels only a finite distance and find that distance.

- **ans:**

(a)

$$v' = -kv, \ v(0) = v_0, \ v = v_0 e^{-kt}$$

$$x' = v, \ x = -\frac{v_0}{k} e^{-kt} + C$$

$$x_0 = -\frac{v_0}{k} + C$$

$$x = x_0 + \frac{v_0}{k}(1 - e^{-kt})$$

(b) $t \to \infty$

$$x \to x_0 + \frac{v_0}{k}$$

2. (2.3:3) Suppose that a motorboat is moving at 40ft/s
24.5 when its motor suddenly quits and that 10 s later the boat has slowed to 20 ft/s. Assume that the resistance it encounters while coasting is proportional to its velocity. How far will the boat coast in all?

- **ans:**

(a)

$$v' = -kv, \ v(0) = v_0 = 40, \ v = v_0 e^{-kt} = 40e^{-kt}$$

$$v(10) = 20, \ 20 = 40e^{-k10}, \ k = \frac{\ln 2}{10}$$

$$x' = v, \ x = -\frac{v_0}{k} e^{-kt} + C$$

$$x_0 = 0 = -\frac{v_0}{k} + C$$

$$x = \frac{v_0}{k}(1 - e^{-kt}) = \frac{40}{0.1 \ln 2}(1 - e^{-0.1 \ln 2t})$$

(b) $t \to \infty$

$$x \to \frac{v_0}{k} = \frac{40}{0.1 \ln 2} = 577$$

3. (2.3:4) Consider a body that moves horizontally
24.7 through a medium whose resistance is proportional to the square of the velocity. Show that

$$v = \frac{v_0}{1 + v_0 kt}$$

and that

$$x(t) = x_0 + \frac{1}{k} \ln(1 + v_0 kt).$$

Find $x(\infty)$.

- **ans:**

$$v' = -kv^2, \ -\int \frac{dv}{v^2} = \int kt$$

$$\frac{1}{v} = kt + C = kt + \frac{1}{v_0}$$

$$v = \frac{v_0}{1 + v_0 kt}$$

$$x = \int v = \frac{1}{k} \ln(1 + v_0 kt) + C$$

$$x = \frac{1}{k} \ln(1 + v_0 kt) + x_0$$

When $t \to \infty$, $x \to \infty$.

4. (2.3:7) Suppose that a car starts from rest, its engine
24.10 providing an acceleration of 10 ft/s^2, while air resistance provides 0.1 ft/s^2 for each foot per second of the car's velocity.
(a) find the car's maximum possible velocity,
(b) Find how long it takes the car to attain 90% of its limiting velocity, and how far it travels while doing so.

- **ans:**

(a)

$$v' = 10 - 0.1v; \quad x(0) = v(0) = 0$$

$$\int \frac{-0.1dv}{10 - 0.1v} = \int (-0.1)dt$$

$$\ln(10 - 0.1v) = -0.1t + C$$

$$\ln(10 - 0.1v) = -0.1t + \ln 10$$

$$v = 100(1 - e^{-0.1t})$$

$$v(\infty) = 100$$

(b)

$$x = \int v = 100t - 1000(1 - e^{-0.1t})$$

90% of $v(\infty)$ is 90.

$$90 = 100(1 - e^{-0.1t})$$

$$t = 23.0259 \quad x(23.0259) = 1402.59$$

5. (2.3:12) It is proposed to dispose of nuclear wastes –
24.15 in drums with weight $W = 640lb$ and volume 8 ft^3 – by dropping them into the ocean. The force equation for a drum falling through water is

$$mv' = -W + B + F_R$$

where the buoyant force B is equal to the weight (at 62.6 lb/ft^3) of the volume of water displaced by the drum and F_R is the force of water resistance, found empirically to be 1 lb for each foot per second. If the drums are likely to burst upon an impact of more than 75ft/s what is the maximum depth to which they can be dropped in the ocean without likelihood of bursting?

- **ans:**

$$\begin{aligned} m &= \frac{640}{32} = 20slugs \\ W &= 640lb \\ B &= 8(62.5) = 500lb \\ F_R &= -v \\ 20v' &= -640 + 500 - v = -140 - v \\ v &= 140(e^{-0.05t} - 1),\ v(0) = 0 \\ v &= -75ft/sec, \quad t = 20 \ln \frac{28}{13} \simeq 15.35s \\ y &= 2800(e^{-0.05t} - 1) - 140t,\ y(0) = 0 \\ y(15.35) &= -648.31 \end{aligned}$$

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1. (3.1:5) Verify that y_1 and y_2 are solutions. Then solve
32.5 the IVP:

$$y'' - 3y' + 2y = 0;\ y_1 = e^x,\ y_2 = e^{2x};\ y(0) = 1,\ y'(0) = 0.$$

- **ans:** Plug in to verify solutions.

$$\begin{aligned} y &= C_1 y_1 + C_2 y_2 \\ 1 &= C_1 + C_2 \\ y' &= C_1 e^x + 2C_2 e^{2x} \\ 0 &= C_1 + 2C_2 \\ y &= 2e^x - e^{2x} \end{aligned}$$

2. (3.1:8) Verify that y_1 and y_2 are solutions. Then solve
32.2 the IVP:

$$y'' - 3y' = 0;\ y_1 = 1,\ y_2 = e^{3x};\ y(0) = 4,\ y'(0) = -2.$$

- **ans:** Plug in to verify solutions.

$$\begin{aligned} 0 - 3(0) &= 0,\ yes \\ 9e^{3x} - 3(3)e^{3x} &= 0,\ yes \end{aligned}$$

The solution is

$$\begin{aligned} y &= C_1 y_1 + C_2 y_2 = C_1 + C_2 e^{3x} \\ 4 &= C_1 + C_2 \\ y' &= 3C_2 e^{3x} \\ -2 &= 3C_2 \\ y &= \frac{14}{3} - \frac{2}{3} e^{3x} \end{aligned}$$

3. (3.1:17) For nonlinear equations or non homgeneous
32.6 equations, a linear combination of solutions may no longer a solution. Show that the given y is a solution, but cy is not a solution:

$$y = \frac{1}{x}; \quad y' + y^2 = 0.$$

- **ans:**

$$y' + y^2 = -\frac{c}{x^2} + \frac{c^2}{x^2} = \frac{c(c-1)}{x^2} = 0$$

So, when $c = 1$, y is a solution. When $c \neq 1$ and $c \neq 0$, cy is not a solution.

4. (3.1:24.) Determine the linear dependence on the real
32.11 line.

$$f(x) = \sin^2 x; \quad g(x) = 1 - \cos 2x;$$

- **ans:** $g(x) = -2\sin^2 x = -2f(x)$ for all x, so they are linearly dependent.

Using Wronskian:

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} \sin^2 x & 1 - \cos 2x \\ 2\sin x \cos x & \sin 2x \end{vmatrix} = 0$$

5. (3.1:22) Determine the linear dependence on the real
32.8 line.

$$f(x) = 1 + x; \quad g(x) = 1 + |x|;$$

- **ans:** When $x < 0$, $g(x) = 1 - x \neq C(1 + x) = f(x)$, they are linearly independent.

When $x > 0$, $g(x) = 1 + x = f(x)$. But we require this to be true for all x in order to claim linearly dependent.

Using Wronskian:

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} 1 + x & 1 + x \\ 1 & 1 \end{vmatrix} = 0$$

6. (3.1:29) Show that the given functions are solutions
32.17 to the IVP. Why it does not contradict the uniqueness theorem?

$$x^2y'' - 4xy' + 6y = 0; \quad y(0) = y'(0) = 0;$$
$$y_1 = x^2;\ y_2 = x^3$$

• **ans:** The uniqueness theorem says when both p and q (and f) are continuous near $x = a$, we have a unique solution to:

$$y'' + p(x)y' + q(x)y = f(x), \quad y(a) = b;\ y'(a) = c$$

Now,

$$p = \frac{-4}{x^2}, \quad q = \frac{6}{x^2}$$

are not continuous at $x = 0$. So the theorem does not guarantee a unique solution. This is why we have two solutions.

7. (3.1:36) Find the general solution:
32.21

$$2y'' + 3y' = 0$$

• **ans:** The characteristic equation gives (by letting $y = e^{rx}$)

$$2r^2 + 3r = 0, \qquad r = 0, -3/2,\ y = C_1 + C_2e^{-3x/2}$$

8. (3.1:40) Find the general solution:
32.23

$$9y'' - 12y' + 4y = 0$$

• **ans:** The characteristic equation gives (by letting $y = e^{rx}$)

$$9r^2 - 12r + 4 = 0, \qquad r = -3/2, -3/2,\ y = (C_1 + C_2x)e^{-3x/2}$$

9. (3.1:41) Find the general solution:
32.25

$$6y'' - 7y' - 20y = 0$$

• **ans:** The characteristic equation gives (by letting $y = e^{rx}$)

$$6r^2 - 7r - 20 = 0,$$
$$(3r + 4)(2r - 5) = 0 \quad r = -4/3, 5/2,\ y = C_1e^{-4x/3} + C_2e^{-5x/2}$$

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1. (3.2:1) Show directly the linearly dependence by find-
34.2 ing a zero linear combination with nonzero coefficients.

$$f(x) = 2x,\ g(x) = 3x^2,\ h(x) = 5x - 8x^2$$

• **ans:** Method 1, (coefficients)

$$A(2x) + B(3x^2) + C(5x - 8x^2) = 0$$

$$A2 + 5C = 0,\ B3 - C8 = 0$$

Let $C = -1$,

$$\frac{5}{2}(2x) + (-\frac{8}{3})(3x^2) + (-1)(5x - 8x^2) = 0$$

Method 2, (choose x, taking derivatives) Let the coefficient of the most complicated term be -1.

$$2Ax + 3Bx^2 = 5x - 8x^2$$

$$\frac{5}{2}(2x) + (-\frac{8}{3})(3x^2) + (-1)(5x - 8x^2) = 0$$

2. (3.2:3) Show directly the linearly dependence by find-
34.3 ing a zero linear combination with nonzero coefficients.

$$f(x) = 0,\ g(x) = \sin x,\ h(x) = e^x$$

• **ans:**

$$A(0) + B(\sin x) + C(e^x) = 0$$

$$(1)(0) + (0)(\sin x) + (0)(e^x) = 0$$

3. (3.2:7) Use the Wronskian to show linearly indepen-
34.6 dence.

$$f(x) = 1,\ g(x) = x,\ h(x) = x^2, \quad \text{the real line}$$

• **ans:**

$$W(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$$

4. (3.2:9) Use the Wronskian to show linearly indepen-
34.4 dence.

$$f(x) = e^x,\ g(x) = \cos x,\ h(x) = \sin x, \quad \text{the real line}$$

• **ans:**

$$W(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = \begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix} = 2e^x \neq 0$$

5. (3.2:15) Given 3 linear linearly independent solutions

34.8
$$y_1 = e^x,\ y_2 = xe^x,\ y_3 = x^2e^x$$

solve the IVP

$$y''' - 3y'' + 3y' - y = 0;\ y(0) = 2,\ y'(0) = 0,\ y''(0) = 0$$

- **ans:**

$$y = C_1y_1 + C_2y_2 + C_3y_3$$

$$\begin{aligned} C_1 + 0 + 0 &= 2 \\ C_1 + C_2 &= 0 \\ C_1 + 2C_2 + 2C_3 &= 0 \end{aligned}$$

$$y = 2e^x - 2xe^x + x^2e^x$$

6. (3.2:21) Given

34.12
$$y_c = c_1 \cos x + c_2 \sin x,\ y_p = 3x$$

solve the IVP

$$y'' + y = 3x;\ y(0) = 2,\ y'(0) = -2$$

- **ans:**

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x + 3x$$

$$\begin{aligned} c_1 &= 2 \\ c_2 + 3 &= -2 \end{aligned}$$

$$y = 2\cos x - 5\sin x + 3x$$

7. (3.2:38) Use the method of variation of parameter to

34.16 solve the DE, given $y_1 = x^3$,

$$x^2y'' + xy' - 9y = 0,\ x > 0$$

- **ans:** We need to find another linearly independent solution y_2. Let

$$y_2 = vy_1$$

We always reduce the order of homogeneous equation by 1.

$$xv'' + 7v' = 0$$

$$\begin{aligned} \frac{dv'}{v'} &= -\frac{7dx}{x} \\ \ln v' &= 7\ln x + \ln A \\ v' &= Ax^{-7} \\ v &= Bx^{-6} + C = x^{-6} \end{aligned}$$

We need only one solution (different from y_1 itself). So we have chosen $B = 1$ and $C = 0$.

$$y_2 = vy_1 = x^{-3}$$

$$y = c_1y_1 + c_2y_2 = c_1x^3 + c_2x^{-3}$$

8. (3.2:39) Use the method of variation of parameter to

34.17 solve the DE, given $y_1 = e^{x/2}$,

$$4y'' - 4y' + y = 0$$

- **ans:** We need to find another linearly independent solution y_2. Let

$$y_2 = vy_1$$

We always reduce the order of homogeneous equation by 1.

$$v'' = 0$$

$$\begin{aligned} v' &= A \\ v &= Ax + B = x \end{aligned}$$

We need only one solution (different from y_1 itself). So we have chosen $A = 1$ and $B = 0$.

$$y_2 = vy_1 = xe^{x/2}$$

$$y = c_1y_1 + c_2y_2 = c_1e^{x/2} + c_2xe^{x/2}$$