M341 H2 (S. Zhang) [b].

1. (1.4:1.) Find the general solution of

8.2

$$\frac{dy}{dx} + 2xy = 0$$

- **ans:** Separable equation!

$$\frac{dy}{y} = -2xdx$$

$$\int \frac{dy}{y} = -\int 2xdx$$

$$\ln y = -x^2 + c$$

$$y = e^{-x^2+c} = Ce^{-x^2}$$

When divided by y above, we lose the singular solution $y = 0$, which is not included in the general solution.

$$y = 0$$

2. (1.4:2.) Find the general solution of

8.3

$$\frac{dy}{dx} + 2xy^2 = 0$$

- **ans:**

$$\int \frac{dy}{y^2} = -\int 2xdx$$

$$-\frac{1}{y} = -x^2 - c$$

$$y = \frac{1}{x^2 + c}$$

When divided by y^2 above, we lose the singular solution $y = 0$, which is not included in the general solution.

$$y = 0$$

3. (1.4:13) Find the general solution of

8.6

$$y^3\frac{dy}{dx} = (y^4 + 1)\cos x$$

- **ans:**

$$\int \frac{y^3 dy}{y^4 + 1} = \int \cos x dx$$

$$\frac{1}{4}\ln(y^4 + 1) = \sin x + C$$

4. (1.4:17.) Find the general solution of

8.7

$$\frac{dy}{dx} = (1 + x)(1 + y)$$

- **ans:**

$$\int \frac{dy}{1 + y} = \int (1 + x)dx$$

$$\ln|1 + y| = x + \frac{1}{2}x^2 + C$$

5. (1.4:24.) Solve the VIP

8.8

$$(\tan y)\frac{dy}{dx} = y; \quad y(\frac{\pi}{2}) = \frac{\pi}{2}$$

- **ans:**

$$\int \frac{dy}{1 + y} = \int \frac{\cos x dx}{\sin x}$$

$$\ln|y| = \ln|\sin x| + c$$

$$y = C\sin x$$

By the initial condition,

$$y = \frac{\pi}{2}\sin x$$

6. (1.4:33.) A certain city had a population of 25000 in
8.17 1960 and a population of 30000 in 1970. Assume that its population will continue to grow exponentially at a constant rate. What population can its city planners expect in the year 2000?

- **ans:**

$$P' = kP, \quad P = P_0 e^{kt}$$

Let $t = 0$ be the year 1960.

$$k = 10\frac{\ln 30000}{\ln 25000} = 0.01823$$

So at year 2000,

$$P(40) = 25000e^{.01823\times 40} = 51840$$

.

Here we do not need to find k. Here is another method.

$$\begin{aligned} P(10) &= P_0 e^{10k} \\ P(40) &= P_0 e^{40k} \\ \frac{P(10)^4}{P(40)} &= \frac{P_0^4}{P_0} \\ P(40) &= \frac{P(10)^4}{P_0^3} = \frac{30000^4}{25000^3} = 51840. \end{aligned}$$

7. (1.4:34.) In a certain culture of bacteria. The number
8.18 of bacteria increased sixfold in 10 h. How long did it take for the population to double?

- **ans:**

$$P' = kP, \quad P = P_0 e^{kt}$$

$$k = \ln(P/P_0)/t = \frac{\ln 6}{10} = 0.17918$$

$$P(t) = 2P_0 = P_0 e^{0.17918t}; t = \frac{\ln 6}{0.17918} = 3.87h$$

.

Here we do not need to find k. Here is another method.

$$\begin{aligned} 6P_0 &= P_0 e^{10k} \\ 2P_0 &= P_0 e^{tk} \\ \frac{6^t}{2^{10}} &= 1 \\ t &= \frac{\ln 2^{10}}{\ln 6} = 3.87h. \end{aligned}$$

8. (1.4:43) A pitcher of buttermilk initially at 25 degree
8.19 is to be cooled by setting it on the front porch, where the temperature is 0 degree. Suppose that the temperature of the buttermilk has dropped 15 degrees after 20 minutes. When will it be at 5 degree.

- **ans:** We have the model, the time rate of change of the temperature of a body immersed in a medium of constant temperature A is proportional to the difference $A - T$:

$$\frac{dT}{dt} = k(A - T)$$

$$\int \frac{dT}{A - T} = \int k dt$$

$$-\ln(A - T) = kt + C,$$

$$(A - T) = ce^{-kt}$$

$$T = A - ce^{-kt}$$

What we know:

$$\begin{aligned} T(0) &= 25 \\ T(20) &= 25 - 15 = 10. \\ A &= 0 \\ T(?) &= 5 \end{aligned}$$

$$T = A - ce^{-kt}$$

$$T = 25e^{-kt}$$

$$10 = 25e^{-k20}$$

$$k = \frac{\ln(10/25)}{-20} = 0.0458$$

$$5 = 25e^{-kt}$$

$$t = \frac{\ln(5/25)}{-k} = 35.1$$

9. (1.4:53) Thousands of years ago ancestors of the Na-
8.23 tive Americans crossed the Bering Strait from Asia and entered the western hemisphere. Since then, the have fanned out across North and South America. The single language that the original Native Americans spoke has since split into many Indian language families. Assume that the number of these language families has been multiplied by 1.5 every 6000 years. There are now 150 Native American language families in the western hemisphere. About when did the ancestors of today's Native Americans arrive?

- **ans:** For the geometry growth model

$$L' = kt$$

$$L(t) = L_0 e^{kt}$$

$$L(6000) = 1.5L_0 \Rightarrow k = \frac{1}{6000} \ln \frac{3}{2}$$

$$L(T) = 150 \Rightarrow 150 = e^{kT}$$

$$T = \frac{1}{k} \ln 150 = 74146$$

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1. (1.5:4) Solve
12.1

$$y' - 2xy = e^{x^2}$$

- **ans:** The integrating factor is

$$\rho = e^{\int -2x} = e^{-x^2}$$

Multiply by the factor, the DE becomes

$$\frac{d}{dx}(e^{-x^2} y) = 1$$

$$e^{-x^2} y = x + c; \ y = (x + c)e^{x^2}$$

2. (1.5:10.) Solve

12.3

$$2xy' - 3y = 9x^2$$

- **ans:** Warning, we must make the coefficient of y' 1:

$$y' - \frac{3}{2x}y = \frac{9}{2}x$$

The integrating factor is

$$\rho = e^{\int -3/(2x)} = e^{-(3/2)\ln x} = x^{-3/2}$$

Multiply by the factor, the DE becomes

$$\frac{d}{dx}(x^{-3/2}y) = \frac{9}{2}x^{1/2}$$

$$x^{-3/2}y = 3x^{3/2} + c;\ y = 3x^3 + cx^{3/2}$$

3. (1.5:12.) Solve

12.5

$$xy' + 3y = 2x^5;\ y(2) = 1$$

- **ans:** Warning, we must make the coefficient of y' 1:

$$y' + \frac{3}{x}y = 2x^4$$

The integrating factor is

$$\rho = e^{\int 3/x} = e^{3\ln x} = x^3$$

Multiply by the factor, the DE becomes

$$\frac{d}{dx}(x^3y) = 2x^7$$

$$x^3y = \frac{1}{4}x^8 + c;$$

$$y = \frac{1}{4}x^5 + cx^{-3}$$

By $y(2) = 1$,

$$y = \frac{1}{4}x^5 - 56x^{-3}$$

4. (1.5:21.) Solve

12.6

$$xy' = 3y + x^4\cos x,\ y(2\pi) = 0$$

- **ans:** Warning, we must normalize the equation:

$$y' + \frac{-3}{x}y = x^3\cos x$$

The integrating factor is

$$\rho = e^{\int -3/x} = e^{-3\ln x} = x^{-3}$$

Multiply by the factor, the DE becomes

$$\frac{d}{dx}(x^{-3}y) = \cos x$$

$$x^{-3}y = \sin x + c;$$

$$y = x^3\sin x + cx^3$$

By $y(2\pi) = 0$,

$$y = x^3\sin x$$

5. (1.5:28) Solve the DE by regarding y as the indepen-
12.8 dent variable.

$$(1 + 2xy)\frac{dy}{dx} = 1 + y^2$$

- **ans:**

Let $x' = dx/dy$.

$$(1 + 2xy) = x'(1 + y^2)$$

Normalize it,

$$x' + \frac{-2y}{1+y^2}x = \frac{1}{1+y^2}$$

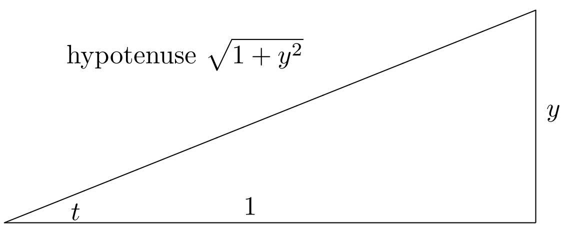
$$\rho = e^{\int \frac{-2y}{1+y^2}dy} = e^{-\ln(1+y^2)} = \frac{1}{1+y^2}$$

Multiply by the factor, the DE becomes

$$\frac{d}{dy}(\frac{1}{1+y^2}x = (1+y^2)^{-2}$$

Let $y = \tan t$

$$\begin{aligned}\int \frac{dy}{(1+y^2)^2} &= \int \frac{\sec^2 t dt}{\sec^4 t} \\ &= \int \cos^2 t dt = \int \frac{1}{2} + \frac{1}{2}\cos 2t \\ &= \frac{t}{2} + \frac{1}{4}\sin 2t \\ &= \frac{t}{2} + \frac{1}{2}\sin t\cos t\end{aligned}$$



By drawing a right triangle of edge length, $y, 1, \sqrt{1+y^2}$ we get

$$\int \frac{dy}{(1+y^2)^2} = \frac{\tan^{-1} y}{2} + \frac{1}{2}\frac{y}{\sqrt{1+y^2}}\frac{1}{\sqrt{1+y^2}}$$

$$\frac{\tan^{-1} y}{2} + \frac{1}{2}\frac{y}{1+y^2}$$

Finally,

$$\frac{1}{1+y^2}x = \frac{\tan^{-1} y}{2} + \frac{1}{2}\frac{y}{1+y^2} + c$$

$$x = (1+y)^2(\frac{\tan^{-1} y}{2} + c) + \frac{y}{2}$$

6. (1.5:38) Consider the cascade of two tanks where the
12.17 output of the first tank is connected to the input of the second tank. Let $V_1 = 100$(gal) and $V_2 = 200$(gal). Each tank contains initially 50lb of salt. The flow rates are all 5 gal/min, which pure water flowing into tank 1.

(a) Find the amount $x(t)$ of salt in tank 1.

(b) Find the amount $y(t)$ of salt in tank 2.

(c) Find the maximum amount of salt ever in tank 2.

- **ans:**

(a) The salt flow-in/out rates are (concentration)(volume flow rate):

$$\text{in}: 0 \cdot 5$$
$$\text{out}: \frac{x(t)}{100}5$$

$$x' = -\frac{x}{20}$$

$$x = x_0 e^{-t/20}$$

$$x = 50e^{-t/20}$$

(b) The salt flow-in/out rates are (concentration)(volume flow rate):

$$\text{in}: \frac{x(t)}{100}5$$
$$\text{out}: \frac{y(t)}{200}5$$

$$y' = \frac{x}{20} - \frac{y}{40}$$
$$y' + \frac{y}{40} = \frac{5}{2}e^{-t/20}$$

1st order linear equation:

$$\mu = e^{\int p} = e^{\int 1/40} = e^{t/40}$$

$$y = \mu^{-1}\int \mu q = e^{-t/40}\int \frac{5}{2}e^{-t/40}$$
$$= e^{-t/40}(-100e^{-t/40} + c)$$

$$y(0) = 50, \Rightarrow y = e^{-t/40}(-100e^{-t/40} + 150)$$

(c) The maximum occurs at a critical point, ie. where the derivative is 0:

$$y = -100e^{-t/20} + 150e^{-t/40}$$
$$y' = 5e^{-t/20} - \frac{15}{4}e^{-t/40}$$
$$= 5e^{-t/20}(1 - \frac{3}{4}e^{t/40}) = 0$$
$$1 = \frac{3}{4}e^{t/40}$$
$$t = 40\ln\frac{4}{3} = 11.51(min)$$
$$y_{max} = y(11.51) = 56.25$$

7. (1.5:41) A 30-year-old woman accepts an engineer-
12.13 ing position with a staring salary of \$30K per year. Here salary $S(t) = 30e^{t/20}$. Meanwhile, 12% of her salary is deposited continuously in a retirement account, which accumulates interest at a continuous annual rate of 6%.

(a) Estimate ΔA in terms of Δt to derive the DE satisfied by the amount $A(t)$ in her retirement account.

(b) Compute $A(40)$, the amount available at her retirement at age 70.

- **ans:**

(a)

$$A' = 0.06A + 0.12S = 0.06A + 0.12(30e^{t/20})$$

$$A' - 0.06A = 3.6e^{t/20}$$

(b)

$$\rho = e^{-0.06t}$$

$$Ae^{-0.06t} = \int 3.6e^{-0.06t}e^{0.05t}$$

$$A = e^{0.06t}(-360e^{-0.01t} + C)$$

$$A(0) = 0$$

$$A = 360(e^{0.06t} - e^{0.05t})$$

$$A(40) \simeq 1,308 \simeq 1.3\text{million}$$

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1. (1.6:5) Solve the differential equation as the type $y' =$
14.4 $f(y/x)$:

$$x(x+y)y' = y(x-y)$$

- **ans:** This is a homogeneous equation. Divided by x on both sides

$$(1 + \frac{y}{x})y' = \frac{y}{x}(1 - \frac{y}{x})$$

We always let

$$v = \frac{y}{x}$$

Then we always have

$$y = xv, \quad y' = v + xv'$$

$$(1+v)(v+xv') = v(1-v)$$

$$(1+v)xv' = -2v^2)$$

Before dividing the equation by v^2, check $v = 0$. $v = 0$ implies $y \equiv 0$, which is a solution. When $v \neq 0$, we get

$$\int (\frac{1}{v^2} + \frac{1}{v})dv = \int -\frac{2}{x}dx$$

$$-\frac{1}{v} + \ln v = -2\ln x + C$$

$$-\frac{x}{y} + \ln y - \ln x = -2\ln x + C$$

$$\ln(xy) = C + \frac{x}{y}$$

2. (1.6:6.) Solve the differential equation as the type
14.5 $y' = f(y/x)$:

$$(x+2y)y' = y$$

• **ans:** This is a homogeneous equation:

$$y' = \frac{y}{x+2y} = \frac{y/x}{1+2(y/x)} = f(\frac{y}{x})$$

We always let

$$v = \frac{y}{x}$$

Then we always have

$$y = xv, \quad y' = v + xv'$$

so (divide the equation by x only)

$$(1+2v)(v+xv') = v$$

$v = 0$ implies $y \equiv 0$, which is a solution. When $v \neq 0$, we get

$$x(2v+1)v' = -2v^2, \quad \int (\frac{2}{v} + \frac{1}{v^2})dv = \int -\frac{2}{x}dx$$

$$\ln v^2 - \frac{1}{v} = -2\ln x + C$$

$$2\ln y - 2\ln x - \frac{x}{y} = -2\ln x + C$$

$$2y\ln y = Cy + x$$

3. (1.6:34) Show the differential equation is exact, and
14.23 solve it as an exact equation.

$$(2xy^2 + 3x^2)dx + (2x^2y + 4y^3)dy = 0$$

• **ans:** (If we can do the second part, we may skip the first part as the solution in the second part implies the equation is exact.)

Exact equation: we write it in the form $F' = 0$:

$$Mdx + Ndy = 0$$

where $F = F(x, y)$, and x and y are independent variable. Then we must have

$$F_x = M, \; F_y = N, \quad M_y = N_x = F_{xy}$$

Here we check

$$M_y = (2xy^2 + 3x^2)_y = 4xy + 0,$$
$$N_x = (2x^2y + 4y^3)_x = 4xy + 0 = M_y$$

Again, note, the above part can be omitted if you can do the second part below correctly. Because the second part, found a solution, implies the equation is exact!

How to find F?

$$F = \int Mdx = \int (2xy^2 + 3x^2)dx$$
$$= x^2y^2 + x^3 + f(y)$$
$$F_y = N$$
$$2x^2y + 0 + f'(y) = 2x^2y + 4y^3, \quad f'(y) = 4y^3$$

So $f = y^4$. Solution of $dF = 0$ is

$$F = C$$

$$x^2y^2 + x^3 + y^4 = C$$

4. (1.6:35) Show the differential equation is exact, and
14.24 solve it as an exact equation.

$$(x^3 + \frac{y}{x})dx + (y^2 + \ln x)dy = 0$$

• **ans:** (If we can do the second part, we may skip the first part as the solution in the second part implies the equation is exact.)

Exact equation: we write it in the form $F' = 0$:

$$Mdx + Ndy = 0$$

where $F = F(x, y)$, and x and y are independent variable. Then we must have

$$F_x = M, \; F_y = N, \quad M_y = N_x = F_{xy}$$

Here we check

$$M_y = (x^3 + \frac{y}{x})_y = 0 + \frac{1}{x},$$
$$N_x = (y^2 + \ln x)_x = 0 + \frac{1}{x} = M_y$$

Note, the above part can be omitted if you can do the second part below correctly. Because the second part, found a solution, implies the equation is exact!

How to find F?

$$F = \int M dx = \int (x^3 + \frac{y}{x}) dx$$
$$= \frac{1}{4}x^4 + y \ln x + f(y)$$
$$F_y = N$$
$$0 + \ln x + f'(y) = y^2 + \ln x, \quad f'(y) = y^2$$

So $f = y^3/3$. Solution of $dF = 0$ is

$$F = C$$

$$\frac{1}{4}x^4 + y \ln x + \frac{1}{3}y^3 = C$$

5. (1.6:36.) Show the differential equation is exact, and
14.14 solve it as an exact equation.

$$(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0$$

- **ans:** Exact equation: we write it in the form $F' = 0$:

$$M dx + N dy = 0$$

where $F = F(x, y)$, and x and y are independent variable. Then we must have

$$F_x = M, \; F_y = N, \quad M_y = N_x = F_{xy}$$

Here we check

$$M_y = (1 + ye^{xy})_y = e^{xy} + xye^{xy},$$
$$N_x = (2y + xe^{xy})_x = e^{xy} + xye^{xy} = M_y$$

Note, the above part can be omitted if you can do the second part below correctly. Because the second part, found a solution, implies the equation is exact!

How to find F?

$$F = \int M dx = \int (1 + ye^{xy}) dx = x + e^{xy} + f(y)$$
$$F_y = xe^{xy} + f'(y) = 2y + xe^{xy}, \quad f'(y) = 2y$$

So $f = y^2$. Solution of $dF = 0$ is

$$F = C$$

$$x + e^{xy} + y^2 = C$$

6. (1.6:46.) Reduce the equation to a first order one,
14.18 and solve it.

$$xy'' + y' = 4x$$

- **ans:** No y! Let $z = y'$

$$xz' + z = 4x$$
$$z' + \frac{1}{x}z = 4$$
$$\mu = e^{\int p} = e^{\ln x} = x$$
$$z = \mu^{-1} \int \mu q = \frac{1}{x} \int 4x = 2x + c/x$$

$$y = \int z = x^2 + c \ln x + D$$

7. (1.6:47.) Reduce the equation to a first order one,
14.16 and solve it.

$$y'' = (y')^2$$

- **ans:** No y! Let $z = y'$

$$z' = z^2$$

If $z = 0$, $y' = 0$, $y = C$. Yes, it is a solution.

If $z \neq 0$,

$$\frac{dz}{z^2} = dx$$

$$-\frac{1}{z} = x + C$$

$$z = -\frac{1}{x + C}$$

$$y = \int -\frac{1}{x + C} dx = -\ln|x + C| + D$$