M242 Study Guide for E final (S. Zhang) .

1. Use Newton's method to find x_3:

2.13

$$x^5 - x - 1 = 0, \; x_1 = 1$$

- **ans:** Find the root for $f(x) = 0$.

$$f(x) = x^5 - x - 1$$

$$f'(x) = 5x^4 - 1$$

Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = 1, \qquad f(x_1) = -1$$

$$f'(x_1) = 4$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.25, \qquad f(x_2) = 0.80$$

$$f'(x_2) = 11.20$$

$$x_3 = 1.1784$$

i	x_i	$f(x_i)$	$f'(x_i)$
1	1	-1	4
2	1.2	0.80	11.20
3	1.1784		

Check by Maple:

The root is 1.167303978 by Maple fsolve.

```
f:=x->x^5-x-1: g:=x->5x^4-1:
t:=1: x1:=t:
evalf([t,f(t),g(t)]); t:=t-f(t)/g(t):
x2:=t:
evalf([t,f(t),g(t)]); t:=t-f(t)/g(t):
x3:=t:
evalf([t,f(t),g(t)]);
fsolve(f(x) = 0, x)
```

2. Find the limit: $\lim_{x\to 0^+} \sin x \ln x$

4.33

- **ans:** It is of form $0 \cdot \infty$. We have to turn one of the function to downstairs.

$$\begin{aligned}
\lim_{x\to 0^+} \sin x \ln x &= \lim_{x\to 0^+} \frac{\ln x}{\csc x} \\
&\overset{\infty/\infty}{=} \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} \\
&= \lim_{x\to 0^+} \frac{\sin x \tan x}{-x} \\
&\overset{0/0}{=} \lim_{x\to 0^+} \frac{\cos x \tan x + \sin x \sec^2 x}{-1} \\
&= \frac{1(0) + 0(1)}{-1} = 0.
\end{aligned}$$

```
limit(sin(x)*ln(x), x =0)
0
```

3. Find the volume of the solid obtained by rotating the region
7.24 bounded by the given cures about the specified line, by
(1) the method of rotation (adding washers/disks),
(2) the method of cylindrical shell (adding thin cylindrical shells).
Sketch the region, and the solid.

$$y = x, \; y = \sqrt{x}, \quad \text{about } y = 1.$$

- **ans:** Find intersections (eliminating y by the two equations):

$$x = \sqrt{x} \Rightarrow x^2 = x$$
$$x = 0, 1 \Rightarrow (0,0), (1,1)$$

(1) by the method of rotation (r is the distance from the curves to $y = 1$)

$$V = \int \pi r^2 h = \int_0^1 \pi((1-x)^2 - (1-\sqrt{x})^2)dx$$
$$= \frac{\pi}{6}$$

(2) by the method of cylindrical shell (r is the distance from a chopped piece to $y = 1$, but h goes from left to right)

$$V == \int 2\pi r h dr = \int_0^1 2\pi(1-y)(x_{right} - x_{left})dy =$$
$$= \int_0^1 2\pi(1-y)(y-y^2)dy = \frac{\pi}{6}$$

```
int((1-x)^2-(1-sqrt(x))^2,x=0..1);
int(2(1-y)*(y-y^2),y=0..1);

plot({x,sqrt(x),1},x=0..1,
scaling=constrained);
```

4. Find $\int e^{2y}\sin y dy$

11.59

• **ans:**

The idea is to make u' "simpler than" u while v' and v are about the "same". But here no matter which one is u, the resulted new integral would be the "same" as the original one. So, we would integration by parts twice to get back the original integral, then solve an equation to get the integral.

$$u = \sin y, \qquad dv = e^{2y}\,dy$$
$$du = \cos y dy, \qquad v = \frac{1}{2}e^{2y}$$

$$\int u dv = uv - \int v du$$

$$\int e^{2y}\sin y dy = \frac{1}{2}e^{2y}\sin y - \frac{1}{2}\int e^{2y}\cos y dy$$

Repeat

$$u = \cos y, \qquad dv = e^{2y}\,dy$$
$$du = -\sin y dy, \qquad v = \frac{1}{2}e^{2y}$$

$$\int e^{2y}\cos y dy = \frac{1}{2}e^{2y}\cos y + \frac{1}{2}\int e^{2y}\sin y dy$$

So

$$\int e^{2y}\sin y dy = \frac{1}{2}e^{2y}\sin y - \frac{1}{4}e^{2y}\cos y - \frac{1}{4}\int e^{2y}\sin y dy$$

That is

$$\frac{5}{4}\int e^{2y}\sin y dy = \frac{1}{2}e^{2y}\sin y - \frac{1}{4}e^{2y}\cos y + c$$

$$\int e^{2y}\sin y dy = \frac{2}{5}e^{2y}\sin y - \frac{1}{5}e^{2y}\cos y + c$$

5. Find $\int \sin^4 x dx$.

13.8

• **ans:** Apply the double angle formula twice here:

$$\int \sin^4 x dx = \int \frac{1}{4}(1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x))dx$$
$$= \frac{1}{4}(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x) + c$$

6. Find $\int \frac{3-x}{(x-1)(x^2-1)}dx$

16.8

• **ans:** We need to use the partial fractions to separate this function.

$$\begin{aligned}&\int \frac{3-x}{(x-1)(x^2-1)}dx\\ =&\int (\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1})dx\\ =&\int (\frac{-1}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x+1})dx\\ =&-(x-1)^{-1} - \ln|x-1| + \ln|x+1| + C\end{aligned}$$

Here, to find out the constants, we let

$$x = 1,\ x = -1,\ x = 0.$$

7. Find $T_{h=2}, T_{h=1}, S_{h=1}$, if

17.28

$$\int_{\sin 8}^{2+\sin 8} f(x)\,dx,\ R_{h=2} = 0,\ L_{h=2} = 512;\ M_{h=2} = 32,$$

• **ans:** It is easy to get the following formulas by the numerical intergration formulas:

$$T_h = \frac{1}{2}R_h + \frac{1}{2}L_h$$

$$T_h = \frac{1}{2}T_{2h} + \frac{1}{2}M_{2h}$$

$$S_h = \frac{1}{3}T_{2h} + \frac{2}{3}M_{2h}$$

$$T_{h=2} = \frac{1}{2}R_{h=2} + \frac{1}{2}L_{h=2} = 256$$

$$T_{h=1} = \frac{1}{2}T_{h=2} + \frac{1}{2}M_{h=2} = 144$$

$$S_{h=1} = \frac{1}{3}T_{h=2} + \frac{2}{3}M_{h=2} = \frac{320}{3}$$

8. Find $\int_{-1}^{2} \frac{x}{\sqrt{|x-1|}}\,dx$ (hint: $\sqrt{|x-1|} = \sqrt{1-x}$ when $x \le 1$.)

18.33

• **ans:** Note there is a singular point in between, $x = 1$. So we have to separate the integral into two.

Both are indefinite integrals (as the function would be infinity at the singular point.)

$$\int_{-1}^{2} \frac{x}{\sqrt{|x-1|}}\,dx = \int_{-1}^{1} \frac{x}{\sqrt{1-x}}\,dx + \int_{1}^{2} \frac{x}{\sqrt{x-1}}\,dx$$

For them, we change variables as follows, respectively.

$$\begin{aligned} u &= \sqrt{1-x} & \qquad w &= \sqrt{x-1} \\ u^2 &= 1-x & w^2 &= x-1 \\ 2udu &= -dx & 2wdw &= dx \end{aligned}$$

$$\begin{aligned} \int_{-1}^{2} \frac{x}{\sqrt{|x-1|}}\,dx &= \int \frac{(1-u^2)(-2udu)}{u} + \int \frac{(w^2+1)2wdw}{w} \\ &= (-2u + \frac{2}{3}u^3) + (\frac{2}{3}w^3 + 2w) \\ &= (-2\sqrt{1-x} + \frac{2}{3}(1-x)^{3/2})_{-1}^{1} \\ &\quad + (\frac{2}{3}(x-1)^3 + 2\sqrt{x-1})_1^2 \\ &= (0 - (-2\sqrt{2} + \frac{2}{3}\sqrt{8})) + ((\frac{2}{3}+2) - 0) \\ &= \frac{2}{3}\sqrt{2} + \frac{8}{3} \end{aligned}$$

9. Test if the following sequence or series converges or diverges
32.62 and show your work (give, at least, the name of the test used). If a sequence or a series converges, find the limit or the sum.

$$\begin{aligned} &(a) \quad 2, \frac{3}{2}, \frac{4}{3}, ..., \frac{n+1}{n}, ... \\ &(b) \quad 2 + \frac{3}{2} + \frac{4}{3} + ... + \frac{n+1}{n} + ... = \sum_{n=1}^{\infty} \frac{n+1}{n} \\ &(c) \quad \frac{1}{2}, \frac{1}{6}, \frac{1}{12}, ..., \frac{1}{n(n+1)}, ... \\ &(d) \quad \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + ... + \frac{1}{n(n+1)} + ... = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \end{aligned}$$

• **ans:** (a) The sequence converges. By L'Hopital rule

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{1+0}{1} = 1$$

(b) The series diverges, by the divergence test, because $a_n \not\to 0$, by (a), $a_n \to 1$:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{1+0}{1} = 1$$

Note that as a sequence, a_n converges, but as a series $\sum a_n$ diverges – adding too many small numbers together we still get an infinity.

(c) The sequence converges.

$$\lim_{n\to\infty} \frac{1}{n(n+1)} = \frac{1}{\infty} = 0$$

(d) The series converges, by the telescoping series computation.

$$\begin{aligned} \frac{1}{n(n+1)} &= \frac{A}{n} + \frac{B}{n+1} \\ &= \frac{1}{n} - \frac{1}{n+1} \\ S_n &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + ... + \frac{1}{n(n+1)} \\ &= \frac{1}{1} - \frac{1}{2} + \\ &\quad \frac{1}{2} - \frac{1}{3} + \\ &\quad \frac{1}{3} - \frac{1}{4} + \\ &\quad \cdots + \frac{1}{n} - \frac{1}{n+1} \\ &= \frac{1}{1} - \frac{1}{n+1} \\ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \lim_{n\to\infty} S_n \\ &= \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1 \end{aligned}$$

Note that this time, the series converges too. We have concellations here. On the other side, the terms are small enough that even we add infinitely many of them we never exceed 1.

10. Write the periodic decimal as a fraction, $1.2\overline{345}$.
32.83 • **ans:** We convert each section of decimal numbers into a fraction. Then we can add fractions together by the formular for geometric series.

$$\begin{aligned} 1.2\overline{345} &= 1.2 + 0.0345(1 + 1000^{-1} + 1000^{-2} + \cdots) \\ &= \frac{12}{10} + \frac{345}{10000}\frac{1}{1-1000^{-1}} \\ &= \frac{11988+345}{9990} = \frac{12333}{9990}. \end{aligned}$$

11. Determine absolute convergence, conditional convergence
36.4 and divergence:

$$(1) \sum \left(\frac{3+2n}{2-n^2}\right)^n, \quad (2) \sum \frac{(-3)^n}{n^3}, \quad (3) \quad \sum \frac{2^n}{n!},$$
$$(4) a_1 = 2,\ a_{n+1} = \frac{2n+1}{n+10} a_n,\ \sum a_n.$$

• **ans:** (1) With the n-th power, it is best to use the root test: (for $n \geq 2$)

$$|a_n|^{1/n} = \left|\frac{3+2n}{2-n^2}\right|^{n \cdot \frac{1}{n}} = \frac{3+2n}{n^2-2}$$
$$\overset{\text{L'Hopital}}{\to} \frac{2}{2n} \to \frac{2}{\infty} = 0 = R < 1$$

By the root test, the series converges absolutely.

(2) We can either use the root test or the ratio test. Using the root test, we need to know the limit of $n^{1/n}$ is 1.

Let us use the ratio test.

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|(-3)^{n+1}/(n+1)^3|}{|(-3)^n/n^3|} = \frac{3}{(1+\frac{1}{n})^3}$$
$$\to \frac{3}{(1+0)^3} = 3 = R > 1$$

The series diverges by the ratio test.

(3) We have to use the ratio test.

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|2^{n+1}/(n+1)!|}{|2^n/n!|} = \frac{2}{n+1}$$
$$\to \frac{2}{\infty} = 0 = R < 1$$

The series converges absolutely by the ratio test.

(4) We do not even know the general form of a_n for the series. So we have to use the ratio test.

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|\frac{2n+1}{n+10}a_n|}{|a_n|} = \frac{2n+1}{n+10}$$
$$= \frac{2+\frac{1}{n}}{1+\frac{10}{n}} \to \frac{2}{1} = 2 = R > 1$$

The series diverges by the ratio test.

12. Find the absolute convergence, conditional convergence
36.31 and divergence by two methods: $\sum_{n=1}^{\infty} 2ne^{-n^2}$

(1) Root test or ratio test;
(2) Integral test or comparison test.

- **ans:** (1) Root test

$$|a_n|^{1/n} = 2^{1/n} n^{1/n} e^{-n} \to 2^0 \cdot 1 \cdot 0 = 0 < 1$$

The series converges absolutely by the root test. Here we may need to use L'H and take log to find

$$\lim_{n\to\infty} n^{1/n} = e^{\lim \frac{\ln n}{n}} = e^{\lim \frac{1/n}{1}} = e^0 = 1.$$

(1') If we use the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2(n+1)e^{-(n+1)^2}}{2ne^{-n^2}} = (1+\frac{1}{n})e^{-2n-1}$$
$$\to 1 \cdot e^{-\infty} = 1 \cdot 0 = 0 < 1$$

The series converges absolutely by the ratio test.

(2) Integral test (you may change variable as $u = -x^2$, but it is easy to see the changing variable by thinking derivative backward)

$$\int_1^{\infty} 2xe^{-x^2}dx = (-e^{-x^2})_1^{\infty} = \frac{1}{e} < \infty$$

Both the series and the integral converge. We have a positive series. So it converges absolutely.

(2') If we use the comparison test:

We compare a given series with either r or p series only. So here we need to find a (bigger) converging series comparible to the given one.

$$2ne^{-n^2} < e^{-n}$$

To prove it, we simplify the inequality to

$$\frac{2ne^{-n^2}}{e^{-n}} < 1$$

$$\frac{2n}{e^{n^2-n}} < 1$$

By L'Hopital rule:

$$\frac{2n}{e^{n^2-n}} \to \frac{2}{(2n-1)e^{n^2-n}} \to \frac{2}{\infty} = 0$$

So, eventually

$$2ne^{-n^2} < e^{-n}$$

Since $\sum e^{-n}$ converges ($r = 1/e < 1$), $\sum 2ne^{-n^2}$ converges too, by the comparison test.

13. Find the radius and interval of convergence:
38.21 $\sum_{n=0}^{\infty}(-1)^n \frac{(x-3)^n}{2n+1}$

- **ans:** We apply the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{|x-3|^{n+1}}{2n+3}}{\frac{|x-3|^n}{2n+1}} = |x-3|\frac{2n+1}{2n+3}$$
$$= |x-3|\frac{2+\frac{1}{n}}{2+\frac{3}{n}} \to |x-3|\frac{2+0}{2+0} = |x-3| < 1$$

comparing to the condition $|x-a| < R \Rightarrow R = 1$

Make sure that the coefficient of x above is 1 when determine R.

Check two end points

$$x = 3-1 = 2, \sum_{n=1}^{\infty} \frac{1}{2n+1} \text{ div }, p = 1.$$

(Here we do not have exactly p series, but it can be compared to the p-series by the limit comparison test.)

$$x = 3+1 = 4, \ \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \text{ conv cond, AST and } p = 1$$

Interval of convergence

$$(2, 4]$$

14. Suppose $\sum_{n=0}^{\infty} c_n(x-1)^n$ converges for $x = 4$ and diverges
38.61 for $x = -3$. What can be said about the convergence of divergence of

$$\sum_{n=0}^{\infty} c_n, \qquad \sum_{n=0}^{\infty} c_n 5^n, \qquad \sum_{n=0}^{\infty} c_n(-3)^n,$$

- **ans:** For a power series: there is a radius R such that:

$$\sum_{n=0}^{\infty} c_n x^n \text{ converges for } |x| < R$$
$$\sum_{n=0}^{\infty} c_n x^n \text{ divverges for } |x| > R$$

From the given conditions, R would be no less than $|x-a| = |4-1| = 3$ but no bigger than $|x-a| = |-3-1| = 4$:

$$R \geq 3, \quad \text{and } R \leq 4.$$

(a) $\sum c_n$: Converge, as $x = 0$, $|x-a| = |0-1| = 1 < 3 \leq R$.

(b) $\sum c_n 5^n$: Diverge, as $x = 6$, $|x-a| = |6-1| = 5 > 4 \geq R$.

(c) $\sum c_n(-3)^n$: Inconclusive, as $x = -2$, $|x-a| = |-2-1| = 3$ which is not smaller than 3 neither bigger than 4.
Note that for $\sum c_n(3)^n$, we would know it converges, as $x = 4$ – the given case.

15. Find the power series (by the geometric series) and the
40.42 Taylor series $f(x) = \ln(1+3x)$, $a = 0$ You need to find the first 3 nonzero terms.

- **ans:** To find the power series representation, we use only the formula for geometric series (we must always start from this formula – absolutely no exception):

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$$

$x \to -3x$

$$\frac{1}{1+3x} = 1 - 3x + 9x^2 - ...$$

Integrate both sides

$$\int \frac{dx}{1+3x} = \int (1 - 3x + 9x^2 - ...)dx$$
$$\frac{1}{3}\ln(1+3x) = x - \frac{3}{2}x^2 + 3x^3 - ... + C$$

Let $x = 0$ to find out C:

$$0 = \frac{1}{3}\ln(1+0) = 0 - 0 + 0 - ... + C$$

Therefore

$$\frac{1}{3}\ln(1+3x) = x - \frac{3}{2}x^2 + 3x^3 - ...$$
$$\ln(1+3x) = 3x - \frac{9}{2}x^2 + 9x^3 - ...$$

The second method – Taylor formula:

$$\begin{aligned} f(x) &= \ln(1+3x), & f(0) &= 0 \\ f'(x) &= 3(1+3x)^{-1}, & f'(0) &= 3 \\ f''(x) &= -9(1+3x)^{-2}, & f''(0) &= -2 \cdot 9 \\ f'''(x) &= 2 \cdot 27(1+3x)^{-3}, & f'''(0) &= 2 \cdot 27 \end{aligned}$$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ...$$
$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + ...$$
$$\ln(1+3x) = 0 + 3x + \frac{-9}{2!}x^2 + \frac{2 \cdot 27}{3!}x^3 + ...$$
$$= 3x - \frac{9}{2}x^2 + 9x^3 - ...$$

16. Find the limit $\lim_{x\to 0} \frac{\sin x - x - x^2}{\cos x - 1}$ by both methods: (1)
40.32 L'Hopital rule, (2) Taylor series expansion.

- **ans:** (1) L'Hopital rule, 0/0.

$$\lim_{x\to 0} \frac{\sin x - x - x^2}{\cos x - 1} \overset{L'H}{=} \lim_{x\to 0} \frac{\cos x - 1 - 2x}{-\sin x}$$
$$\overset{L'H}{=} \lim_{x\to 0} \frac{-\sin x - 2}{-\cos x}$$
$$= \frac{-2}{-1} = 2$$

(2) Taylor series expansion. We need to know the formula:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$$

$$\lim_{x\to 0}\frac{\sin x - x - x^2}{\cos x - 1}$$
$$= \lim_{x\to 0}\frac{(x-\frac{x^3}{3!}+\frac{x^5}{5!}-...)-x-x^2}{(1-\frac{x^2}{2!}+\frac{x^4}{4!}-...)-1}$$
$$= \lim_{x\to 0}\frac{-x^2+...}{-\frac{x^2}{2!}+...}$$

(divide by x^2 – note that the higher order terms are all zero)

$$= \lim_{x\to 0}\frac{-1+...}{-\frac{1}{2!}-...} = \frac{-1}{-\frac{1}{2!}} = 2$$

17. Find the Taylor polynomial $T_3(x)$ for $f(x) = \ln(1+2x)$, $a = 1$, $x \in [0.5, 1.5]$. And bound the error for $T_3(x)$ by R_3 on the given interval.

41.42

- **ans:** Computing derivatives at $a = 1$.

$$f(x) = \ln(1+2x), \qquad f(1) = \ln 3$$
$$f'(x) = 2(1+2x)^{-1}, \qquad f'(a) = \frac{2}{3}$$
$$f''(x) = -4(1+2x)^{-2}, \qquad f''(a) = -\frac{4}{9}$$
$$f'''(x) = 16(1+2x)^{-3}, \qquad f''(a) = \frac{16}{27}$$

The last term is for R_3:

$$\frac{f^{(4)}(z)}{4!} = -4(1+2z)^{-4}$$

$$T_3 = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$
$$= \ln 3 + \frac{2}{3}(x-1) - \frac{2}{9}(x-1)^2 + \frac{8}{81}(x-1)^3$$
$$R_3(x) = \frac{f^{(4)}(z)}{4!}(x-a)^4 = -4(1+2z)^{-4}(x-1)^4$$

To board the error, we find the worst case, i.e., z is the smallest and x is the biggest:

$$\max_{x\in[0.5,1.5]} |R_3(x)| \le 4(1+2\cdot 0.5)^{-4}|1.5-1|^3$$
$$= 0.015625$$

Rough checking by a calculator:

$$f(0.5) - T_3(0.5) = -0.00423$$
$$f(1.5) - T_3(1.5) = -0.00244$$

18. Given $\begin{cases} x = 1+3t^2 \\ y = 4+2t^3, \end{cases} \quad 0 \le t \le 1$

52.61

(1) Find dy/dx and d^2y/dx^2 at $t = 1$.

(2) Find an equation of the line tangent at $t = 1$.

(3) Find the area of the region that lies under the curve.

(4) Find the arc length.

- **ans:** (1)

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{6t^2}{6t} = t$$

$$t = 1, \qquad \frac{dy}{dx} = 1$$

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}}{dx} = \frac{(t)'}{x'} = \frac{1}{6t}$$

$$t = 1, \qquad \frac{d^2y}{dx^2} = \frac{1}{6}$$

(2) Find the point,

$$t = 1, \qquad x = 4, \ y = 6$$

The slope is dy/dx at $t = 1$.

$$\frac{y-6}{x-4} = 1$$

(3) As in regular graph $y = y(x)$, we have

$$A = \int y\,dx = \int_0^1 yx'\,dt$$
$$= \int_0^1 (4+2t^3)(0+6t)dt = \int_0^1 (24t+12t^4)dt$$
$$= 12 + \frac{12}{5} = \frac{72}{5}$$

(4) arc length:

$$L = \int \sqrt{dx^2+dy^2} = \int \sqrt{(x')^2+(y')^2}dt$$
$$= \int_0^1 \sqrt{(6t)^2+(6t^2)^2}dt$$
$$= 6\int_0^1 t\sqrt{1+t^2}dt = 6\int \sqrt{1+t^2}\frac{1}{2}d(1+t^2)$$
$$= 3\left.\frac{(1+t^2)^{3/2}}{3/2}\right|_0^1 = 2(2\sqrt{2}-1)$$

19. Fill:

53.21

$(x,\ y)$	(r, θ)
$(3, -3)$	$(\ \ ,\ \ )$
$(\ \ , 2)$	$(\ \ , -\pi/2)$

- **ans:**

$$r^2 = x^2 + y^2, \ \tan\theta = y/x$$

(1)

$$r = \pm\sqrt{x^2+y^2} = \pm 3\sqrt{2}$$

$$\tan\theta = y/x = -1$$

$(3\sqrt{2}, -\pi/4)_{r\theta}$; or $(-3\sqrt{2}, 3\pi/4)_{r\theta}$

(2)

$$y = r\sin\theta \;\Rightarrow 2 = r\sin(-\pi/2)$$

So $r = -2$.

$$x = r\cos\theta \;\Rightarrow x = -2\sin(-\pi/2) = 0$$

20. (1)Sketch the curves (must show the data table for the
54.6 points),
(2) find the intersection points (in (r,θ) coordinates), and
(3) find the area of the region inside both $r = 2+2\cos\theta$ and $r = 2$.

- **ans:**

θ	0	$\pi/4$	$\pi/2$	...
$r_1 = 2$	2	2	2	...
$r_2 = 2+2\cos\theta$	4	$2+\sqrt{2}$	2	...

Intersections: $\cos\theta = 0$, $\theta = -\pi/2, \pi/2$. Two pieces of common areas: one is a piece of pie, on the left side of y axis. The other piece has two small pieces, but can be represented by one integrals.

$$A = \int \frac{1}{2} r^2 d\theta$$

```
 int(2^2/2,t=-Pi/2..Pi/2)
 +int((2+2cos(t))^2/2,t=Pi/2..3Pi/2);
int((2+2cos(t))^2/2,t);
```

$$\int_{\pi/2}^{3\pi/2} (2+2\cos\theta)^2 d\theta = 3\theta + 4\sin\theta + \frac{1}{2}\sin 2\theta$$

$$\begin{aligned} A =& \frac{1}{2}\int_{-\pi/2}^{\pi/2} 2^2 d\theta + \frac{1}{2}\int_{\pi/2}^{3\pi/2} (2+2\cos\theta)^2 d\theta \\ &= 5\pi - 8 \end{aligned}$$

21. Find the equation for the parabola, and sketch:
55.6 vertex $(3,2)$, focus $(3,6)$

- **ans:** Focus is above vertex, $p = 4$.

Directrix is distance p on the other side of vertex:

$$y = -2$$

$$(x-3)^2 = 4p(y-2) = 16(y-2)$$

22. Find the equation of an ellipse, then sketch:
55.11 foci $(0,2), (0,6)$, vertices $(0,0)$, $(0,8)$

- **ans:** center $(0,4)$. $a = 4$, $c = 2$.

$$b = \sqrt{a^2 - c^2} = \sqrt{12}$$

$$\frac{(x-0)^2}{12}+\frac{(y-4)^2}{4^2}=1$$

```
rx:=x=-sqrt(12)..sqrt(12):
plot({[x,4+4*sqrt(1-x^2/12),rx],
      [x,4-4*sqrt(1-x^2/12),rx],
 [x,8,rx], [x,0,rx],
 [-sqrt(12),y,y=0..8],
 [ sqrt(12),y,y=0..8]
 },scaling=constrained);
```

```
 plot({[x,-1+2sqrt(1+x^2),x=-2..2],
       [x,-1-2sqrt(1+x^2),x=-2..2],
[x,-1+2x,x=-2..2],[x,-1-2x,x=-2..2],
  [x,-3,x=-1..1], [x,1,x=-1..1],
  [-1,y,y=-3..1],[1,y,y=-3..1]
  },scaling=constrained);
```

23. Identify the type of conic section and find the vertices and
55.16 foci, and sketch: $y^2+2y=4x^2+3$

- **ans:**

$$(y+1)^2-4x^2=4$$

$$-\frac{(x)^2}{1^2}+\frac{(y+1)^2}{2^2}=1,$$

vertices $(0,-1\pm 2)$, $c=\sqrt{5}$, foci $(0,-1\pm\sqrt{5})$.

$$-\frac{(x)^2}{1^2}+\frac{(y+1)^2}{2^2}=0, y=-1\pm 2x$$