M242 Study Guide for E 3 (S. Zhang) .

1. Find the convergence by the integral test.

33.23

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$$

• **ans:** We find the integral is convergent: (integration by parts)

$$\begin{aligned}
&\int_1^\infty \frac{\ln n dn}{n^3} \\
\overset{u=\ln n,\ dv=dn/n^3}{\underset{du=dn/n,\ v=-n^{-2}/2}{=}} & -\frac{n^{-2}}{2}\ln n + \int \frac{n^{-2}}{2}\frac{dn}{n} \\
=& -\frac{\ln n}{2n^2} + \frac{1}{2}\int \frac{dn}{n^3} \\
=& -\frac{\ln n}{2n^2} - \frac{1}{4}n^{-2} + c \\
=& \left(-\frac{\ln n}{2n^2} - \frac{1}{4}n^{-2}\right)_1^\infty \\
=& (0-0) - (0-\frac{1}{4}) = \frac{1}{4}
\end{aligned}$$

Note that, using L'Hopital rule, we have

$$\lim_{n\to\infty} \frac{\ln n}{n^2} = \lim_{n\to\infty} \frac{1/n}{2n} = \frac{0}{\infty} = 0$$

The series is convergent by the integral test.

2. (1) Show the convergence by the integral test.

33.33 (2) Find the sum by the telescoping series method.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6}$$

• **ans:** (1)

$$\begin{aligned}
x^2+5x+6 &= (x+2)(x+3) \\
\frac{1}{x^2+5x+6} &= \frac{A}{x+2} + \frac{B}{x+3}
\end{aligned}$$

$$\begin{aligned}
\int_1^\infty \frac{dx}{x^2+5x+6} =& \int \frac{2dx}{(x+1)(x+3)} \\
=& \int (\frac{1}{x+2} - \frac{1}{x+3})dx \\
=& \ln \frac{x+2}{x+3}|_1^\infty = -\ln\frac{3}{4} = 0.287
\end{aligned}$$

Yes, both the integral and series converge.

(2)

$$\frac{1}{n^2+5n+6} = \frac{1}{n+2} - \frac{1}{n+3}$$

$$\begin{aligned}
s_n =& \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \dots + \frac{1}{n+2} - \frac{1}{n+3} \\
=& \frac{1}{3} - \frac{1}{n+3} \\
\to & \frac{1}{3} = 0.3333
\end{aligned}$$

Yes, as the function is decreasing, the area of rectangles is a little bigger than the area under the curve (the right point rule).

3. Find the convergence by comparison test:

34.4

$$\sum_{n=2}^{\infty} \frac{3+\sin n}{n\ln n}$$

• **ans:** In comparison test, to show a series diverges, we need to find a smaller series (termwise) diverges too.

By the integral test, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ diverges, because the integral diverges (you may make variable substitution $u = \ln x$):

$$\int_2^\infty \frac{dx}{x\ln x} = (\ln(\ln x))_2^\infty = \infty$$

We compare the given series with this series

$$\frac{3+\sin n}{n\ln n} \geq \frac{3-1}{n\ln n} > \frac{1}{n\ln n}$$

So, by the comparison test, (the bigger series)

$$\sum_{n=2}^{\infty} \frac{3+\sin n}{n\ln n} \text{ diverges.}$$

4. Find the convergence:

35.12

$$\sum_{n=2}^{\infty} (-1)^n \sin\frac{1}{\ln n}$$

• **ans:** Yes, $b_n \searrow 0$, the series converges by AST. To show $b_n \searrow 0$, we use the derivative and the limit:

$$\begin{aligned}
b_n &= \sin\frac{1}{\ln n} \\
f(n) &= \sin\frac{1}{\ln n} > 0, \text{ if } n > 2 \\
f'(n) &= -\left(\cos\frac{1}{\ln n}\right)(\ln n)^{-2}\frac{1}{n} < 0, \text{ if } n > 2. \\
\lim_{n\to\infty} b_n &= \sin\frac{1}{\infty} = \sin 0 = 0.
\end{aligned}$$

5. Show the series is convergent. How many terms of the
35.14 series do we need to add in order to find the sum to the indicated accuracy?

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^8},\ |\text{error}| < 10^{-4}$$

- **ans:**

$$\begin{aligned} b_n &= \frac{1}{n^8} \\ f(n) &= \frac{1}{n^8} > 0, \\ f'(n) &= -n^{-9} = -\frac{1}{n^9} < 0 \\ b_n &\searrow 0 \end{aligned}$$

Yes, $b_n \searrow 0$, the series converges by AST.

Now, by the error bound for the alternating series, we get

$$\begin{aligned} R_n &< b_{n+1} \\ b_{n+1} &< 10^{-4} \\ \frac{1}{(n+1)^8} &< 10^{-4} \\ n+1 &> 10^{1/2} = 3.1622 \\ n &> 2. \end{aligned}$$

We compute the first 3 terms.

6. Determine absolute convergence, conditional convergence
36.4 and divergence:

$$\begin{aligned} &(1) \sum\left(\frac{3+2n}{2-n^2}\right)^n \\ &(2) \sum\frac{(-3)^n}{n^3} \\ &(3) \sum\frac{2^n}{n!} \\ &(4) a_1 = 2,\ a_{n+1} = \frac{2n+1}{n+10}a_n,\ \sum a_n \end{aligned}$$

- **ans:** (1) With the n-th power, it is best to use the root test: (for $n \geq 2$)

$$\begin{aligned} |a_n|^{1/n} &= \left|\frac{3+2n}{2-n^2}\right|^{n\cdot\frac{1}{n}} = \frac{3+2n}{n^2-2} \\ &\overset{\text{L'Hopital}}{\to} \frac{2}{2n} \to \frac{2}{\infty} = 0 = R < 1 \end{aligned}$$

By the root test, the series converges absolutely.

(2) We can either use the root test or the ratio test. Using the root test, we need to know the limit of $n^{1/n}$ is 1.

Let us use the ratio test.

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|(-3)^{n+1}/(n+1)^3|}{|(-3)^n/n^3|} = \frac{3}{(1+\frac{1}{n})^3} \\ &\to \frac{3}{(1+0)^3} = 3 = R > 1 \end{aligned}$$

The series diverges by the ratio test.

(3) We have to use the ratio test.

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|2^{n+1}/(n+1)!|}{|2^n/n!|} = \frac{2}{n+1} \\ &\to \frac{2}{\infty} = 0 = R < 1 \end{aligned}$$

The series converges absolutely by the ratio test.

(4) We do not even know the general form of a_n for the series. So we have to use the ratio test.

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|\frac{2n+1}{n+10}a_n|}{|a_n|} = \frac{2n+1}{n+10} \\ &= \frac{2+\frac{1}{n}}{1+\frac{10}{n}} \to \frac{2}{1} = 2 = R > 1 \end{aligned}$$

The series diverges by the ratio test.

7. Find the absolute convergence, conditional convergence
36.32 and divergence by two methods:

$$\sum_{n=1}^{\infty} 2^{n+2}$$

(a) Root test or ratio test;

(b) Integral test or comparison test, or divergence test.

- **ans:** (1) Root test

$$|a_n|^{1/n} = 2\cdot 2^{2/n} \to 2\cdot 2^0 = 2 > 1$$

The series diverges by the root test.

If we use the ratio test: +

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+3}}{2^{n+2}} = 2 > 1$$

The series diverges by the ratio test.

(2) Integral test

$$\int_1^{\infty} 2^{x+2}dx = \left(\frac{2^{x+2}}{\ln 2}\right)_1^{\infty} = \infty$$

Both the series and the integral diverge.

If we use the comparison test, we need to compare it with an r-series (itself is an r-series too)

$$2^{n+2} > 2^n$$

$\sum 2^n$ diverges ($r = 2 > 1$, r-test). By the comparison test, (the bigger one) $\sum 2^{n+2}$ diverges.

If we use the divergence test,

$$a_n = 2^{n+2} \to 2^\infty = \infty \neq 0$$

By the divergence test, $\sum 2^{n+2}$ diverges.

8. Find the radius of convergence, interval of convergence:

38.21

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$$

- **ans:** We apply the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{|x-3|^{n+1}}{2n+3}}{\frac{|x-3|^n}{2n+1}} = |x-3| \frac{2n+1}{2n+3}$$
$$= |x-3| \frac{2+\frac{1}{n}}{2+\frac{3}{n}} \to |x-3| \frac{2+0}{2+0} = |x-3| < 1$$

comparing to the condition $|x-a| < R \;\Rightarrow\; R = 1$

Check two end points

$$x = 3-1 = 2, \; \sum_{n=1}^{\infty} \frac{1}{2n+1} \text{ div } , p = 1.$$

(Here we do not have exactly p series, but it can be compared to the p-series by the limit comparison test.)

$$x = 3+1 = 4, \; \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \text{ conv cond, AST and } p = 1$$

Interval of convergence

$$(2, 4]$$

9. Suppose $\sum_{n=0}^{\infty} c_n x^n$ converges for $x = -4$ and diverges for $x = 6$. What can be said about the convergence about

38.23

$$(a) \; \sum_{n=0}^{\infty} c_n$$
$$(b) \; \sum_{n=0}^{\infty} c_n 8^n$$
$$(c) \; \sum_{n=0}^{\infty} c_n (-3)^n$$
$$(d) \; \sum_{n=0}^{\infty} (-1)^n c_n 9^n$$

- **ans:** For power series: there is a radius R such that:

$$\sum_{n=0}^{\infty} c_n x^n \text{ converges for } |x| < R$$
$$\sum_{n=0}^{\infty} c_n x^n \text{ divverges for } |x| > R$$

From the given conditions, R would be no less than 4 but no bigger than 6:

$$R \geq 4, \quad \text{and } R \leq 6.$$

(a) Converge, as $x = 0 < 4 \leq R$.
(b) Diverge, as $x = 8 > 6 \geq R$.
(c) Converge, as $x = -3, |-3| < 4 \leq R$.
(d) Diverge, as $x = -9, |-9| > 6 \geq R$.

10. Find the power series representation and its radius of convergence

39.23

$$f(x) = \frac{x^2}{(1-2x)^2}$$

- **ans:** We note that the x^2 on the top does not matter as we can pull it out, or multiply the series by it.

Note that

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$$

we differentiate both sides of

$$\frac{1}{1-x} = 1 + x + x^2 + ...$$

to get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + ... + (n+1)x^n + ...$$

Replace x by $2x$: to get

$$\frac{1}{(1-2x)^2} = 1 + 2(2x) + 3(2x)^2 + ... + (n+1)(2x)^n + ...$$

Multiply the series by x^2

$$\frac{x^2}{(1-2x)^2} = x^2 + 2(2x^3) + ... + (n+1)2^n x^{n+2}$$
$$= \sum_{n=0}^{\infty} (n+1) 2^n x^{n+2}$$

Ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+2)2^{n+1}x^{n+3}}{(n+1)2^n x^{n+2}} = \frac{1+\frac{2}{n}}{1+\frac{1}{n}} 2|x|$$
$$\to 2|x| < 1$$
$$|x| < \frac{1}{2}$$

Radius of convergence $R = 1/2$.

Done.

If we find the interval of convergence, we continue to check two end points: $x = -1/2$

$$\sum_{n=0}^{\infty}(n+1)2^n x^{n+2} = \sum_{n=0}^{\infty}(n+1)(-1)^n \frac{1}{4}$$

The series diverges by the divergence test ($a_n \not\to 0$)

The other end point $x = 1/2$.

$$\sum_{n=0}^{\infty}(n+1)2^n x^{n+2} = \sum_{n=0}^{\infty}(n+1)\frac{1}{4}$$

The seies diverges by the divergence test($a_n \not\to 0$)

The interval of convergence:

$$(-\frac{1}{2}, \frac{1}{2})$$

11. Find the power series (by the geometric series) and the
40.4 Taylor series

$$f(x) = \ln(1+x),\ a = 0$$

and use it to evaluate

$$1 - \frac{1}{2} + \frac{1}{3} - ...$$

• **ans:** To find the power series representation, we use only the formula for geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + ...$$

$x \to -x$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + ...$$

Integrate both sides

$$\int \frac{dx}{1+x} = \int (1 - x + x^2 - x^3 + ...)dx$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + C$$

Let $x = 0$ to find out C:

$$0 = \ln(1+0) = 0 - 0 + 0 - ... + C$$

Therefore

$$\begin{aligned}\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}\end{aligned}$$

The second method – Taylor formula:

$$\begin{aligned}f(x) &= \ln(1+x), & f(0) &= 0\\ f'(x) &= (1+x)^{-1}, & f'(0) &= 1\\ f''(x) &= -(1+x)^{-2}, & f''(0) &= -1\\ f'''(x) &= 2(1+x)^{-3}, & f'''(0) &= 2\end{aligned}$$

$$\begin{aligned}&f^{(n)}(x) = (-1)^{n-1}[(n-1)!](1+x)^{-n},\\ &f^{(n)}(0) = (-1)^{n-1}[(n-1)!]\end{aligned}$$

$$\begin{aligned}f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ...\\ &+ \frac{f^{(n)}(a)}{n!}(x-a)^n + ...\\ \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + (-1)^{n-1}\frac{x^n}{n} + ...\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}\end{aligned}$$

Let $x = 1$ in the power series representation for the series:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - ...$$

12. Find the Taylor series
40.51

$$f(x) = \sin x,\ a = \frac{\pi}{2}$$

• **ans:**

$$\begin{aligned}f(x) &= \sin x, & f(\frac{\pi}{2}) &= 1\\ f'(x) &= \cos x, & f'(\frac{\pi}{2}) &= 0\\ f''(x) &= -\sin x, & f''(\frac{\pi}{2}) &= -1\\ f'''(x) &= -\cos x, & f(a) &= 0\end{aligned}$$

Then the computation repeats.

$$\begin{aligned}\sin x =& f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ...\\ =& 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!}\\ &- \frac{(x-\frac{\pi}{2})^6}{6!} + ...\end{aligned}$$

Or

$$\sum_{n=0}^{\infty}(-1)^n\frac{(x-\frac{\pi}{2})^{2n}}{(2n)!}$$

13. Use a well known Maclaurin series to find the Maclaurin
40.53 series for

$$f(x) = e^x + e^{2x}$$

• **ans:**

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \\ e^{2x} &= 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + ... \\ e^x + e^{2x} &= 2 + 3x + \frac{5x^2}{2!} \\ &+ ... + \left(\frac{x^n}{n!} + \frac{(2x)^n}{n!} \right) + ... \\ &= 2 + 3x + \frac{5x^2}{2!} \\ &+ ... + \frac{(1+2^n)x^n}{n!} + ... \end{aligned}$$

Or

$$e^x + e^{2x} = \sum_{n=0}^{\infty} \frac{(1+2^n)x^n}{n!}$$

14. Find the limit by both methods: (1) L'Hopital rule, (2)
40.31 Taylor series expansion.

$$\lim_{x\to 0} \frac{\sin x^2}{\cos x - 1}$$

• **ans:** (1) 0/0.

$$\begin{aligned} \lim_{x\to 0} \frac{\sin x^2}{\cos x - 1} &\overset{L'H}{=} \lim_{x\to 0} \frac{2x \cos x^2}{-\sin x} \\ &\overset{L'H}{=} \lim_{x\to 0} \frac{2\cos x^2 - 4x \sin x^2}{-\cos x} \\ &= \frac{2-0}{-1} = -2 \end{aligned}$$

(2)

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... \\ \sin x^2 &= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - ... \end{aligned}$$

$$\begin{aligned} &\lim_{x\to 0} \frac{\sin x^2}{\cos x - 1} \\ &= \lim_{x\to 0} \frac{x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - ...}{(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...) - 1} \\ &= \lim_{x\to 0} \frac{x^2 + ...}{-\frac{x^2}{2!} + ...} \\ &= \lim_{x\to 0} \frac{1 + ...}{-\frac{1}{2!} - ...} = \frac{1}{-\frac{1}{2!}} = -2 \end{aligned}$$

15. Find the Taylor polynomial $T_2(x)$ for the function f at
41.4 the number a. And bound the error for $T_2(x)$ by R_2 on the given interval .

$$f(x) = \frac{1}{\sqrt{x}}, \ a = 1, \ x \in [0.9, 1.2]$$

• **ans:**

$$\begin{aligned} f(x) &= x^{-1/2}, & f(1) &= 1 \\ f'(x) &= -\frac{1}{2}x^{-3/2}, & f'(a) &= -\frac{1}{2} \\ f''(x) &= \frac{3}{4}x^{-5/2}, & f''(a) &= \frac{3}{4} \\ f'''(x) &= -\frac{15}{8}x^{-7/2}, \end{aligned}$$

$$\begin{aligned} T_2 &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &= 1 - \frac{1}{2}(x-1) + \frac{3}{8}(x-1)^2 \\ R_2(x) &= \frac{f'''(z)}{3!}(x-a)^3 = -\frac{15/8}{3!}z^{-7/2}(x-1)^3 \end{aligned}$$

$$\begin{aligned} \max_{x\in[0.9,1.2]} |R_2(x)| &= \max_{x\in[0.9,1.2]} \frac{15}{8\cdot 3!} \frac{1}{|z|^{7/2}} |x-1|^3 \\ &\le \frac{15}{8\cdot 3!} \frac{1}{0.9^{7/2}} |1.2-1|^3 \\ &= \frac{5}{16} \cdot \frac{0.2^3}{\sqrt{0.9^7}} = 0.00963 \end{aligned}$$

Rough checking:

$$\begin{aligned} f(0.9) - T(0.9) &= \frac{1}{\sqrt{0.9}} - (1 - \frac{1}{2}(-0.1) + \frac{3}{8}(0.01)) \\ &= 0.00034 \\ f(1.2) - T(1.2) &= \frac{1}{\sqrt{1.2}} - (1 - \frac{1}{2}(0.2) + \frac{3}{8}(0.04)) \\ &= -0.00212 \end{aligned}$$