M242 Study Guide for E 2 (S. Zhang) .

1. Find $\int \sin^5 x \cos^2 x dx$.

   - **ans:** $u = \cos x$

$$\int \sin^5 x \cos^2 x dx = \int (1-u^2)^2 u^2 (-du)$$
$$= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + c$$

2. Find $\int \sin^4 x dx$.

   - **ans:**

$$\int \sin^4 x dx = \int \frac{1}{4}(1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x))dx$$
$$= \frac{1}{4}(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x) + c$$

3. Find $\int \sin 4x \cos 5x dx$.

   - **ans:**

$$\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$$
$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \sin(A+B))$$
$$\cos A \cos B = \frac{1}{2}(\cos(A-B) + \sin(A+B))$$

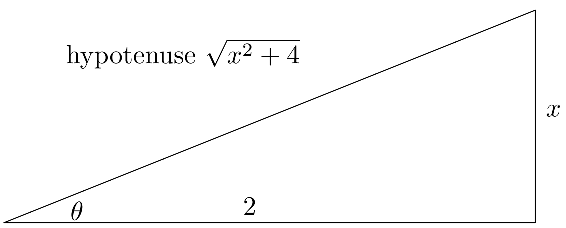
$$\int \sin 4x \cos 5x dx = \frac{1}{2}\int (\sin(-x) + \sin 9x)dx$$
$$= \frac{1}{2}(\cos x - \frac{1}{9}\cos 9x) + C$$

4. Find $\int \frac{dx}{x^2\sqrt{x^2+4}}$

   - **ans:**

$$x = 2\tan\theta$$

$$\int = \int \frac{2\sec^2\theta}{4\tan^2\theta(2\sec\theta)}d\theta$$
$$= \int \frac{\cos\theta}{\sin^2\theta}d\theta$$
$$= -\frac{1}{4\sin\theta} + C$$



$$\sin\theta = \frac{x}{\sqrt{x^2+4}}$$

$$\int = -\frac{\sqrt{x^2+4}}{4x} + C$$

Check the answer by taking derivative.

5. Find $\int \frac{x^2}{x^2-4x+5}dx$

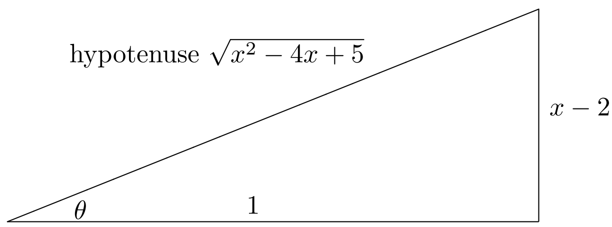
   - **ans:**

$$x^2 - 4x + 5 = x^2 - 4x + 4 + 1 = (x-2)^2 + 1$$

$$\frac{x^2}{x^2-4x+5} = \frac{x^2}{(x-2)^2+1}$$

$$x - 2 = \tan\theta$$

$$\int = \int \frac{(\tan\theta+2)^2\sec^2\theta}{\sec^2\theta}d\theta$$
$$= \int (\sec^2\theta + 4\frac{\sin\theta}{\cos\theta} + 3)d\theta$$
$$= \tan\theta - 4\ln\cos\theta + 3\theta + C$$



$$\tan\theta = \frac{x-2}{1}$$

$$\int = (x-2) + 2\ln|x^2 - 4x + 5|$$
$$+ 3\tan^{-1}(x-2) + C$$

Check the answer by taking derivative.

6. Find $\int \frac{x^2+1}{(x^2-2x+2)^2}dx$

   - **ans:**

$$x^2 - 2x + 2 = (x-1)^2 + 1$$

It is of type $\sqrt{x^2+a^2}$. Here we can do a change of variable first $u = x - 1$. But it is better to expand the idea above, letting

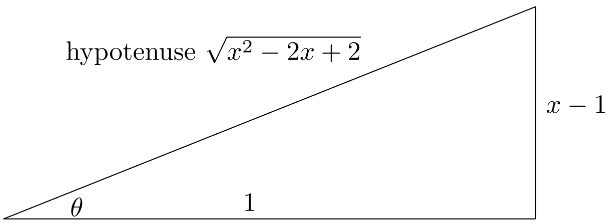
$$x - 1 = \tan\theta$$

$$dx = \sec^2\theta d\theta$$

$$\begin{aligned} x^2+1 &= (1+\tan\theta)^2+1 \\ &= 1+2\tan\theta+\tan^2\theta+1 \\ &= 2\tan\theta+\sec^2\theta+1 \end{aligned}$$

$$\begin{aligned} &\frac{x^2+1}{(x^2-2x+2)^2}dx \\ &= \int \frac{2\tan\theta+\sec^2\theta+1}{\sec^4\theta}\sec^2\theta d\theta \\ &= \int (2\sin\theta\cos\theta+1+\cos^2\theta)d\theta \\ &= \int (\sin 2\theta+\frac{3}{2}+\frac{1}{2}\cos 2\theta)d\theta \\ &= -\frac{1}{2}\cos 2\theta+\frac{3}{2}\theta+\frac{1}{4}\sin 2\theta+C \\ &= \sin^2\theta+\frac{3}{2}\theta+\frac{1}{2}\sin\theta\cos\theta+C_1 \end{aligned}$$

To find θ and $\cos\theta$, we draw a right triangle:



$$\begin{aligned} &\frac{x^2+1}{(x^2-2x+2)^2}dx \\ &= \frac{(x-1)^2}{x^2-2x+2}+\frac{3}{2}\tan^{-1}(x-1)+\frac{1}{2}\frac{(x-1)}{x^2-2x+2}+C \\ &= \frac{3}{2}\tan^{-1}(x-1)+\frac{1}{2}\frac{(x-3)}{x^2-2x+2}+C \end{aligned}$$

7. Find

(1) $\displaystyle\int \frac{3x-8}{x^2-5x+6}dx$

(2) $\displaystyle\int \frac{2x^3-10x^2+15x-8}{x^2-5x+6}dx$

- **ans:** (1)

$$x^2-5x+6 = (x-2)(x-3)$$

$$\frac{3x-8}{x^2-5x+6} = \frac{A}{x-3}+\frac{B}{x-2}$$

$$3x-8 = A(x-2)+B(x-3)$$

$$\begin{aligned} x &= 2, 6-8 = B \\ x &= 3, 9-8 = A \end{aligned}$$

$$\begin{aligned} \int \frac{3x-8}{x^2-5x+6}dx &= \int \frac{1}{x-3}dx+\int \frac{2}{x-2}dx \\ &= \ln|x-3|+2\ln|x-2|+C \end{aligned}$$

(2) Division:

$$\frac{2x^3-10x^2+15x-8}{x^2-5x+6} = 2x+\frac{3x-8}{x^2-5x+6}$$

$$\begin{aligned} &\int \frac{2x^3-10x^2+15x-8}{x^2-5x+6}dx \\ &= \int 2xdx+\int \frac{1}{x-3}dx+\int \frac{2}{x-2}dx \\ &= x^2+\ln|x-3|+2\ln|x-2|+C \end{aligned}$$

8. Find $\displaystyle\int \frac{3-x}{(x-1)(x^2-1)}dx$

- **ans:**

$$\begin{aligned} &\int \frac{3-x}{(x-1)(x^2-1)}dx \\ &= \int (\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1})dx \\ &= \int (\frac{-1}{x-1}+\frac{1}{(x-1)^2}+\frac{1}{x+1})dx \\ &= -(x-1)^{-1}-\ln|x-1|+\ln|x+1|+C \end{aligned}$$

Here, to find out the constants, we let

$$x = 1, \ x = -1, \ x = 0.$$

9. Write out the partial fraction forms for $f(x)$. But do not determine the numerical values the coefficients A, B, C etc. For example, $\dfrac{2}{x(x-1)} = \dfrac{A}{x}+\dfrac{B}{x-1}$, that is all.

(1) $\dfrac{x^2+x+2}{x(x-2)(x^2-1)}$,

(2) $\dfrac{5x+1}{x(x^2+2x)(x^2+2)^2}$

- **ans:**

$$\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1}+\frac{D}{x-2}$$

$$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2}+\frac{Dx+F}{x^2+2}+\frac{Gx+H}{(x^2+2)^2}$$

10. Find $\displaystyle\int_1^\infty \frac{dx}{x^2(1+x^2)}$

- **ans: Method 1.** Par. frac. $\dfrac{1}{x^2(1+x^2)} = \dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{Cx+D}{1+x^2} = \dfrac{1}{x^2}-\dfrac{1}{1+x^2}$; $\displaystyle\int = \left[-\frac{1}{x}-\tan^{-1}x\right]_1^\infty = (0-\frac{\pi}{2})-(-1-\frac{\pi}{4}) = 1-\frac{\pi}{4}$.

Method 2. Trig. sub. $x = \tan u, dx = \sec^2 u du$; $\int = \int \frac{\sec^2 u du}{\tan^2 u \sec^2 u} = \int \cot^2 u du = \int \csc^2 u - 1 du = \cot u - u = \frac{1}{\tan u} - u = \left[\frac{1}{x} - \tan^{-1} x\right]_1^\infty = 1 - \frac{\pi}{4}$.

11. Compute by the trapezoidal rule with $h = 3$ and $h = 1.5$ (2 methods) and find the error bound each time $\int_1^4 8x^2\, dx$

- **ans:** Exact solution is $(8x^3/3)_1^4 = 168$.

$$h = 3, \quad x_0 = 1, x_1 = 4,\ n = 1$$

$$\begin{aligned} T_{h=3} &= h(\frac{1}{2}f(x_0) + \frac{1}{2}f(x_1)) \\ &= 204 \end{aligned}$$

$$K_2 = \max_{a \le x \le b} |f''| = 16(1) = 16$$

$$\begin{aligned} |E_T| &\le K_2 \frac{(b-a)^3}{12n^2} \\ &= 16(3)^3/(12 \cdot 1^2) = 36 \end{aligned}$$

Actural error is $168 - 204 = -36$. So we can roughly see the above calculations of the integral and the error are right.

$$h = 1.5, \quad x_0 = 1, x_1 = 2.5, x_2 = 4,\ n = 2$$

$$\begin{aligned} T_{h=1.5} &= \frac{3}{2}(\frac{1}{2}f(x_0) + f(x_1) + \frac{1}{2}f(x_2)) \\ &= 177 \end{aligned}$$

$$\begin{aligned} |E_T| &\le K_2 \frac{(b-a)^3}{12n^2} \\ &= 16(3)^3/(12 \cdot 2^2) = 9 \end{aligned}$$

Actual error is $168 - 177 = -9$.So we can roughly see the above calculations of the integral and the error are right.

Method 2: (Save more function evalutaions than the mid-point rule)

$$\begin{aligned} T_{h=1.5} &= \frac{1}{2}T_{h=3} + h(f(2.5)) \\ &= 204/2 + (3/2)(2(5)^2) = 354/2 = 177. \end{aligned}$$

12. Given

$$\int_{\sin 8}^{2+\sin 8} f(x)\, dx,\ R_{h=2} = 0,\ L_{h=2} = 512;\ M_{h=2} = 32,$$

Find $T_{h=2}, T_{h=1}, S_{h=1}$.

- **ans:**

$$T_h = \frac{1}{2}R_h + \frac{1}{2}L_h$$

$$T_h = \frac{1}{2}T_{2h} + \frac{1}{2}M_{2h}$$

$$S_h = \frac{1}{3}T_{2h} + \frac{2}{3}M_{2h}$$

$$T_{h=2} = \frac{1}{2}R_{h=2} + \frac{1}{2}L_{h=2} = 256$$

$$T_{h=1} = \frac{1}{2}T_{h=2} + \frac{1}{2}M_{h=2} = 144$$

$$S_{h=1} = \frac{1}{3}T_{h=2} + \frac{2}{3}M_{h=2} = \frac{320}{3}$$

13. Find

$$\int_0^{35} \frac{1}{\sqrt[3]{x-8}}\, dx$$

- **ans:**

$$\int_0^{35} = \int_0^8 \frac{1}{-(8-x)^{1/3}}\, dx + \int_8^{35} \frac{1}{(x-8)^{1/3}}\, dx$$

Let

$$\begin{aligned} u &= (8-x)^{1/3} & w &= (x-8)^{1/3} \\ u^3 &= 8 - x & w^3 &= x - 8 \\ 3u^2 du &= -dx & 3w^2 dw &= dx \end{aligned}$$

$$\begin{aligned} \int_0^{35} &= -\int \frac{(-3u^2 du)}{u} + \int \frac{3w^2 dw}{w} \\ &= (\frac{3}{2}u^2) + (\frac{3}{2}w^2) \\ &= (\frac{3}{2}(8-x)^{2/3})_0^8 + (\frac{3}{2}(x-8)^{2/3})_8^{35} \\ &= (0 - (\frac{3}{2}2^2)) + (\frac{3}{2}3^2 - 0) \\ &= \frac{15}{2} \end{aligned}$$

14. Find convergence:

$$\int_2^\infty \frac{x}{x^6+1} dx$$

- **ans:** ideas: Compare to some known integrals.

To show convergence, compare with something bigger and convergent.

To show divergence, compare with something smaller and divergent.

Try something not working:

$$\int_2^\infty \frac{x}{x^2+1}dx$$
$$\int_2^\infty \frac{x}{x^4+1}dx$$

Method 1

$$\frac{x}{x^6+1} < \frac{x}{x^6}$$

$$\int_2^\infty \frac{x}{x^6}dx = (x^{-4}/(-4))_2^\infty$$
$$= \frac{1}{64} < \infty$$

The bigger one converges $\Rightarrow$ the smaller one converges.

Method 2

$$\frac{x}{x^6+1} < \frac{3x^2}{x^6+1}$$

$$\int_2^\infty \frac{3x^2}{x^6+1}dx = (\tan^{-1} x^3)_2^\infty$$
$$= \frac{\pi}{2} - \tan^{-1} 8 < \infty$$

The bigger one converges $\Rightarrow$ the smaller one converges.

15. Find

$$\int \frac{3x^2+x+4}{x^4+3x^2+2}dx$$

- **ans:**

$$x^4+3x^2+2 = (x^2+1)(x^2+2)$$

$$\frac{3x^2+x+4}{x^4+3x^2+2} = \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+2}$$

$$\begin{aligned}&3x^2+x+4\\ =&(Bx^3+Cx^2+2Bx+2C)+(Dx^3+Ex^2+Dx+E)\end{aligned}$$

$$\begin{aligned}B+D&=0\\ C+E&=3\\ 2B+D&=1\\ 2C+E&=4\end{aligned}$$

$$B=1,\ C=1,\ D=-1,\ E=2$$

$$\begin{aligned}&\int \frac{3x^2+x+4}{x^4+3x^2+2}dx\\ =&\int (\frac{x}{x^2+1} + \frac{1}{x^2+1} - \frac{x}{x^2+2} + \frac{1}{x^2+2})dx\\ =&\frac{1}{2}\ln(x^2+1) + \tan^{-1} x\\ &-\frac{1}{2}\ln(x^2+2) + \sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}} + C\end{aligned}$$

16. Find

$$\int_{-1}^2 \frac{2}{\sqrt{|x|}}\,dx$$

(hint: $\sqrt{|x|} = \sqrt{-x}$ when $x \le 0$.)

- **ans:**

$$\begin{aligned}\int_{-1}^2 \frac{2}{\sqrt{|x|}}\,dx &= \int_{-1}^0 \frac{2}{\sqrt{|x|}}\,dx + \int_0^2 \frac{2}{\sqrt{|x|}}\,dx\\ &= \int_{-1}^0 \frac{2dx}{\sqrt{-x}} + \int_0^2 \frac{2dx}{\sqrt{-x}}\end{aligned}$$

Let

$$\begin{aligned}u&=\sqrt{-x} & w&=\sqrt{x}\\ u^2&=-x & w^2&=x\\ 2udu&=-dx & 2wdw&=dx\end{aligned}$$

$$\begin{aligned}\int_{-1}^2 \frac{2}{\sqrt{|x|}}\,dx &= \int \frac{-4udu}{u} + \int \frac{4wdw}{w}\\ &= -4u+4w = (-4\sqrt{-x})_{-1}^0 + (4\sqrt{w})_0^2\\ &= (0-(-4)) + (4\sqrt{2}-0) = 4+4\sqrt{2}\end{aligned}$$

17. Find the arc length on the curve from $(0,-1)$ to $(0,1)$:

$$x^2+y^2=1$$

- **ans:** (Old method)

$$L = \frac{1}{2}s = \pi r = \pi$$

(New method) Two pieces of curves

$$\begin{aligned}y&=\sqrt{1-x^2}, & y&=-\sqrt{1-x^2},\\ y'&=\frac{-x}{\sqrt{1-x^2}}, & y&=\frac{x}{\sqrt{1-x^2}}\end{aligned}$$

For both pieces:

$$\sqrt{1+(y')^2} = (1-x^2)^{-1/2}$$

$$
\begin{aligned}
L &= \int_a^b \sqrt{1+(y')^2}dx \\
&= \int_0^1 (1-x^2)^{-1/2}dx + \int_0^1 (1-x^2)^{-1/2}dx
\end{aligned}
$$

$$
\begin{aligned}
&\overset{x=\sin\theta}{=} 2\int d\theta \\
&= 2\sin^{-1} x|_0^1 = 2(\frac{\pi}{2} - 0) = \pi
\end{aligned}
$$

(New method 2)

$$x = \sqrt{1-y^2} \quad \sqrt{1+(x')^2} \qquad = \frac{1}{\sqrt{1-y^2}}$$

$$
\begin{aligned}
L &= \int_a^b \sqrt{1+(x')^2}dy \\
&= \int_{-1}^1 \frac{1}{\sqrt{1-y^2}}dy \\
&\overset{y=\sin\theta}{=} \int d\theta \\
&= \sin^{-1} y|_{-1}^1 = \pi
\end{aligned}
$$

18. Find arc length:

$$y^2 = 4(x+4)^3,\ y > 0,\ 0 \le x \le 2.$$

- **ans:** Taking square root and choose the positive sign:

$$
\begin{aligned}
y &= 2(x+4)^{3/2} \\
y' &= 3(x+4)^{1/2} \\
\sqrt{1+y'^2} &= \sqrt{1+9(x+4)} \\
&= \sqrt{9x+37}
\end{aligned}
$$

$$
\begin{aligned}
L &= \int_a^b \sqrt{1+(y')^2}dx \\
&= \int_0^2 \sqrt{9x+37}dx \\
&= \frac{(9x+37)^{3/2}}{(3/2)(9)}\Big|_0^2 \\
&= \frac{2}{37}(55^{3/2} - 37^{3/2}) \\
&= 13.54286690
\end{aligned}
$$

From the function graph, the y range goes from about 15 to 30. So the arc length is roughly 15. Our answer is pretty close.

19. Show, by the sandwich theorem,

$$
\begin{aligned}
&(1)\ \lim \frac{(-1)^2}{\ln(n+1)} = 0 \\
&(2)\ \lim \frac{\sin n}{n} = 0 \\
&(3)\ \lim \frac{3^n}{n!} = 0
\end{aligned}
$$

- **ans:** (1)

$$|a_n| = \frac{1}{\ln(n+1)} \to 0$$

(2)

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$$

(3)

$$0 \le \frac{3^n}{n!} \le \frac{3}{1}\frac{3}{2}\cdots\frac{3}{n} \le \frac{9}{2}\cdot\frac{3}{n} \to 0$$

20. We can show (not required) that the sequence

$$
\begin{aligned}
a_1 &= 1 \\
a_{n+1} &= \frac{12}{7-a_n}, \qquad n = 1, 2, ...
\end{aligned}
$$

is increasing and bounded above. Therefore it is convergent. Find the limit.

- **ans:** Take limit on the equation

$$
\begin{aligned}
L &= \frac{12}{7-L} \\
7L - L^2 &= 12 \\
L^2 - 7L + 12 &=, \quad L = 3, 4
\end{aligned}
$$

To find out the unique limit, we check the first few terms:

$$1,\ 2,\ \frac{12}{5},\ \frac{60}{23}, ...$$

The sequence is increasing, but less than 3.

The correct answer for the limit is 3.

Another way to find the answer. Since the sequence starts from 1 and increases, as it has a stationary point at 3, the sequence won't pass 3 to reach the other stationary point 4.

21. Test if the following sequence or series converges or diverges and show your work (give, at least, the name of the test used). If a sequence or a series converges, find the limit or the sum.

$(a)\quad 2, \frac{3}{2}, \frac{4}{3}, ..., \frac{n+1}{n}, ...$

$(b)\quad 2 + \frac{3}{2} + \frac{4}{3} + ... + \frac{n+1}{n} + ... = \sum_{n=1}^{\infty} \frac{n+1}{n}$

$(c)\quad \frac{1}{2}, \frac{1}{6}, \frac{1}{12}, ..., \frac{1}{n(n+1)}, ...$

$(d)\quad \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + ... + \frac{1}{n(n+1)} + ... = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

- **ans:** (a) The sequence converges. By L'Hopital rule

$$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{1+0}{1} = 1$$

(b) The series diverges, by the divergence test, because $a_n \not\to 0$, by (a), $a_n \to 1$.

(c) The sequence converges.

$$\lim_{n\to\infty} \frac{1}{n(n+1)} = \frac{1}{\infty} = 0$$

(d) The series converges, by the telescoping series computation.

$$\begin{aligned}
\frac{1}{n(n+1)} &= \frac{A}{n} + \frac{B}{n+1} \\
&= \frac{1}{n} - \frac{1}{n+1} \\
S_n &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + ... + \frac{1}{n(n+1)} \\
&= \frac{1}{1} - \frac{1}{2} + \\
&\frac{1}{2} - \frac{1}{3} + \\
&\frac{1}{3} - \frac{1}{4} + \\
&\cdots + \frac{1}{n} - \frac{1}{n+1} \\
&= \frac{1}{1} - \frac{1}{n+1} \\
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \lim_{n\to\infty} S_n \\
&= \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1
\end{aligned}$$

22. Find the sum of the series: $\sum_{n=1}^{\infty} \frac{4}{(n+2)(n+4)}$

- **ans:** This is a telescoping series. To find the sum, we need to rewrite the rational functions as partial fractions: $\frac{4}{(n+2)(n+4)} = \frac{A}{(n+2)} + \frac{B}{(n+4)} = \frac{2}{(n+2)} + \frac{-2}{(n+4)}$. The partial sum would be $\sum_{n=0}^{N} \frac{4}{(n+2)(n+4)} = \left(\frac{2}{3} - \frac{2}{5}\right) + \left(\frac{2}{4} - \frac{2}{6}\right) + \left(\frac{2}{5} - \frac{2}{7}\right) + \left(\frac{2}{6} - \frac{2}{8}\right) + \cdots + \left(\frac{2}{N+2} - \frac{2}{N+4}\right) = \frac{2}{3} + \frac{2}{4} - \frac{2}{N+2} - \frac{2}{N+4}$. When N goes to infinity, the partial sum is the whole sum of the series. Since the last two terms go to zero when N goes to infinity, the sum of the series is $\frac{2}{3} + \frac{2}{4}$. **Note** that you are not asked to test if the series converges or not here. So, you do not need to apply tests like the integral test, the direct or comparison tests here (though they can tell the series converges, but they could not tell what the limit the series converges to – the sum).

23. Find the sum of the series: $\sum_{n=0}^{\infty} \frac{3-2^n}{5^n}$.

- **ans:** There are two geometric series combined here. We need to separate it into two. $\sum_{n=0}^{\infty} \frac{3-2^n}{5^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{5}\right)^n - \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = \frac{3}{1-(1/5)} - \frac{1}{1-(2/5)} = \frac{25}{12}$.

24. Write the periodic decimal as a fraction, $1.2\overline{345}$.

- **ans:** $1.2 + 0.0345(1 + 1000^{-1} + 1000^{-2} + \cdots) = \frac{12}{10} + \frac{345}{10000} \frac{1}{1-1000^{-1}} = \frac{11988+345}{9990} = \frac{12333}{9990}$