M242 Study Guide for E 1 (S. Zhang) .

1. Use Newton's method to find x_3:

2.13

$$x^5 - x - 1 = 0,\ x_1 = 1$$

- **ans:**

$$f(x) = x^5 - x - 1$$

$$f'(x) = 5x^4 - 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\begin{aligned} x_1 &= 1, & f(x_1) &= -1 \\ & & f'(x_1) &= 4 \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1.25, & f(x_2) &= 0.80 \\ & & f'(x_2) &= 11.20 \\ x_3 &= 1.1784 \end{aligned}$$

x_i	$f(x_i)$	$f'(x_i)$
1	−1	4
1.2	0.80	11.20
1.1784		

Check by Maple:

The root is 1.167303978 by Maple fsolve.

```
f:=x->x^5-x-1: g:=x->5x^4-1:
t:=1: x1:=t:
evalf([t,f(t),g(t)]); t:=t-f(t)/g(t):
x2:=t:
evalf([t,f(t),g(t)]); t:=t-f(t)/g(t):
x3:=t:
evalf([t,f(t),g(t)]);
fsolve(f(x) = 0, x)
```

2. Find the limit

4.26

$$(1)\ \lim_{x\to\infty}\left(1-\frac{2}{x}\right)^{4x}$$

$$(2)\ \lim_{x\to 1}\left(1-\frac{2}{x}\right)^{4x}$$

- **ans:**

```
limit((1-2/x)^(4*x), x =infinity)
exp(-8)
```

(1) It is of type 1^∞. Before doing it, we know the answer is between 0 and 1 since the base is less than 1.

$$F = \left(1-\frac{2}{x}\right)^{4x}$$

$$\begin{aligned} \lim_{x\to\infty}\ln F &= \lim_{x\to\infty}\frac{4\ln(1-2/x)}{1/x} \\ &= \lim_{x\to\infty}\frac{4(1-2/x)^{-1}(2x^{-2})}{-x^{-2}} \\ &= \lim_{x\to\infty}\frac{4(-2)}{1-2/x} \\ &= -8 \end{aligned}$$

$$\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^{4x} = e^{\lim_{x\to\infty}\ln F} = e^{-8}$$

(2) It is not an indeterminant form:

$$\lim_{x\to 1}\left(1-\frac{2}{x}\right)^{4x} = (-1)^4 = 1$$

3. Find the limit

4.32

$$\lim_{x\to 0}\frac{\sin 4x}{\tan 5x}$$

- **ans:** 0/0, using L'Hospital rule.

$$\begin{aligned} \lim_{x\to 0}\frac{\sin 4x}{\tan 5x} &\overset{0/0}{=} \lim_{x\to 0}\frac{(\sin 4x)'}{(\tan 5x)'} \\ &= \lim_{x\to 0}\frac{4\cos 4x}{5\sec^2 5x} = \frac{4\cdot 1}{5\cdot 1} = \frac{4}{5} \end{aligned}$$

```
limit(sin(4x)/tan(5x), x =0)
4/5
```

4. Find the limit

4.33

$$\lim_{x\to 0^+}\sin x\ln x$$

- **ans:** It is of form $0\cdot\infty$. We have to turn one of the function to downstairs.

$$\begin{aligned} \lim_{x\to 0^+}\sin x\ln x &= \lim_{x\to 0^+}\frac{\ln x}{\csc x} \\ &\overset{\infty/\infty}{=} \lim_{x\to 0^+}\frac{1/x}{-\csc x\cot x} \\ &= \lim_{x\to 0^+}\frac{\sin x\tan x}{-x} \\ &\overset{0/0}{=} \lim_{x\to 0^+}\frac{\cos x\tan x + \sin x\sec^2 x}{-1} \\ &= \frac{1(0)+0(1)}{-1} = 0. \end{aligned}$$

```
limit(sin(x)*ln(x), x =0)
0
```

5. Find the limit

4.35

$$\lim_{x\to\infty}(x - \ln x)$$

- **ans:** It is of form $\infty - \infty$. Usually, we to combine them to get $0/0$ or ∞/∞.

$$\lim_{x\to\infty}(x - \ln x) = \lim_{x\to\infty} x(1 - \frac{\ln x}{x})$$

When we combine them, we need to know the

$$\lim_{x\to\infty}\frac{\ln x}{x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty}\frac{1/x}{1} = 0$$

So

$$\lim_{x\to\infty}(x - \ln x) = \lim_{x\to\infty} x(1 - \frac{\ln x}{x}) = \infty \cdot 1 = \infty$$

If we are not careful, we would get this error:

$$\begin{aligned}\lim_{x\to\infty}(x - \ln x) &= \lim_{x\to\infty} x(1 - \frac{\ln x}{x})\\ &= \lim_{x\to\infty}\frac{1 - \frac{\ln x}{x}}{\frac{1}{x}}\\ &\overset{\frac{\infty}{\infty}}{\neq} \lim_{x\to\infty}\frac{0 - \frac{1-\ln x}{x^2}}{-\frac{1}{x^2}}\\ &= \lim_{x\to\infty}(1 - \ln x)\\ &= 1 - \infty = -\infty \text{ Wrong!}\end{aligned}$$

```
limit(x*ln(x), x =infinity)
infinity
```

6. Sketch the region. Find the area of the region bounded by the curves. 5.12

$$y = x^2 - 2x,\ y = x + 4$$

- **ans:** Find the intersections of the two curves:

$$\begin{aligned}x^2 - 2x &= x + 4\\ x^2 - 3x - 4 = 0,\ (x-4)(x+1) &= 0\\ x = -1, 4,\ \Rightarrow (-1,3),\ &(4,8)\end{aligned}$$

From the graph, we can see it is much simpler to find the area by narrow vertical rectangles, i.e. integrating against x:

$$\begin{aligned}A &= \int y_{top} - y_{bottom}\\ &= \int_{-1}^{4}((x+4) - (x^2 - 2x))dx =\\ &= \left(-\frac{1}{3}x^3 + \frac{3}{2}x^2 + 4x\right)_{-1}^{4}\\ &= \frac{125}{6} \sim 21\end{aligned}$$

Note that the area of the rectangle containing this region is $5 \times 9 = 45$. So the answer here is reasonable.

```
plot({x^2-2x,x+4},x=-1..4,
scaling=constrained);
int((x^2-2x)-(x+4),x);
int((x^2-2x)-(x+4),x=-1..4);
```

7. Sketch the region. Find the area. 5.26

$$\int_1^4 |\sqrt{2x+3} - x| dx$$

- **ans:** Find the intersection(s) of the two curves:

$$\begin{aligned}y = \sqrt{2x+3} &= x\\ x^2 - 2x - 3 &= 0\\ x = -1, 3,\ \Rightarrow &(3,3)\end{aligned}$$

The other x is outside the integral.

From the graph, we can determine how to choose a sign for the absolute function. But we can "cheat" by computing both integrals and use positive areas for them.

$$
\begin{aligned}
A_1 =&\int_1^3 (\sqrt{2x+3}-x)dx\\
=&\left(\frac{1}{3}(2x+3)^{3/2}-\frac{1}{2}x^2\right)_1^3\\
=&(9-\frac{9}{2})-(\frac{5}{3}\sqrt{5}-\frac{1}{2})
\end{aligned}
$$

$$
\begin{aligned}
A_2 =&\int_3^4 (\sqrt{2x+3}-x)dx\\
=&\left(\frac{1}{3}(2x+3)^{3/2}-\frac{1}{2}x^2\right)_3^4\\
=&(\frac{11}{3}\sqrt{11}-4)
\end{aligned}
$$

So

$$A=|A_1|+|A_2|=A_1-A_2.$$

```
plot([[x,x,x=1..4],[x,sqrt(2x+3),x=1..4],
[4,y,y=sqrt(11)..4],[1,y,y=1..sqrt(5)],
[0,y,y=0..0.1]],scaling=constrained);
int(sqrt(2*x+3)-x,x=1..3);
int(sqrt(2*x+3)-x,x=3..4);
int(sqrt(2*x+3)-x,x)
```

8. 6.11 Find the volume of the solid obtained by rotating the region bounded by the given cures about the specified line. Sketch the region, and the solid.

$$y=x,\ y=\sqrt{x},\ \text{ about } y=1.$$

- **<u>ans:</u>** Find intersections:

$$x=x^2\Rightarrow x=0,1\Rightarrow(0,0),(1,1)$$

We cut the solid of rotation vertically, into thin washers: (note that the radius is the distance for each curve to the the line $y=1$)

$$
\begin{aligned}
V=&\int A(x)dx=\int\pi(r_{out}^2-r_{in}^2)\\
&\int_0^1\pi((1-x)^2-(1-\sqrt{x})^2)dx\\
=&\frac{\pi}{6}
\end{aligned}
$$

Rough checking: here, the volune of cone would be

$$\frac{1}{3}\pi(1)^2(1)=\frac{1}{3}\pi.$$

So the above result might be right, though it seems a little too big.

```
int((1-x)^2-(1-sqrt(x))^2,x=0..1);
int(2y*(y-y^2),y=0..1);

plot({x,sqrt(x),1},x=0..1,
scaling=constrained);
```

9. 7.18 Find the volume of the solid obtained by rotating the region bounded by the given cures about the given line.

$$y=x^2,\ y=2-x^2;\quad \text{ about } x=1\ .$$

- **<u>ans:</u>** We sketch the region first. It would help us to decide which way to cut the solid is better.

Intersection points:

$$y=x^2,\ y=2-x^2,\Rightarrow x^2=2-x^2$$

$$x^2=1\Rightarrow x=\pm1\Rightarrow(-1,1),(1,1)$$

If we cutthe solid horizontally into stack of thin washers, then there are two integrals. The top integral is from the curve $y=2-x^2$ to itself. And the bottom integral is from the curve $y=x^2$ to itself. It would be easier to cut the solid vertically as thin cylindrical shells.

Height of a shell is

$$h = y_{top} - y_{bottom} = (2 - x^2) - x^2$$

Radius of the cylindrical shell is from a x on the shell to the center $x = 1$:

$$r = (1) - 1 = 1 - x$$

$$\begin{aligned} V &= \int 2\pi r h dx \\ &= \int_{-1}^{1} 2\pi(1-x)(2-2x^2)dx \\ &= 2\pi(\frac{1}{2}x^4 - \frac{2}{3}x^3 - x^2 + 2x)_{-1}^{1} \\ &= \frac{16}{3}\pi \sim 5\pi \end{aligned}$$

Rough checking: Should be smaller than the outside cylinder:

$$\begin{aligned} V_c &= \pi(r^2)h \\ &= \pi(2^2)2 = 8\pi. \end{aligned}$$

```
plot([[1,t,t=0..2],[x,x^2,
 x=-1..1],[x,2-x^2,x=-1..1]],
scaling=constrained);
int( (1-x)*( 2-2x^2),x );
int( (1-x)*( 2-2x^2),x=-1..1);
```

10. 7.24 Find the volume of the solid obtained by rotating the region bounded by the given cures about the specified line, by
(1) the method of rotation (adding washers/disks),
(2) the method of cylindrical shell (adding thin cylindrical shells).
Sketch the region, and the solid.

$$y = x, \ y = \sqrt{x}, \quad \text{about } y = 1.$$

- **ans:** Find intersections:

$$\begin{aligned} x = \sqrt{x} &\Rightarrow x^2 = x \\ x = 0, 1 &\Rightarrow (0,0), (1,1) \end{aligned}$$

(1) by the method of rotation (r is the distance from the curves to $y = 1$)

$$\begin{aligned} V &= \int \pi r^2 h = \int_0^1 \pi((1-x)^2 - (1-\sqrt{x})^2)dx \\ &= \frac{\pi}{6} \end{aligned}$$

(2) by the method of cylindrical shell (r is the distance from a chopped piece to $y = 1$, but h goes from left to right)

$$\begin{aligned} V == \int 2\pi r h dr = \int_0^1 2\pi(1-y)(x_{right} - x_{left})dy = \\ = \int_0^1 2\pi(1-y)(y-y^2)dy = \frac{\pi}{6} \end{aligned}$$

```
int((1-x)^2-(1-sqrt(x))^2,x=0..1);
int(2(1-y)*(y-y^2),y=0..1);

plot({x,sqrt(x),1},x=0..1,
scaling=constrained);
```

11. 9.18 Find the average value and a c so that $f(c) = f_{ave}$:

$$f(x) = (x-2)^2, \ x \in [2, 5]$$

Draw a graph with a rectangle of height $f(c)$.

- **ans:**

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

$$\begin{aligned} &= \frac{1}{5-2}\int_2^5 (x-2)^2 dx \\ &= \frac{1}{3}\left(\frac{1}{3}x^3 - 2x^2 + 4x\right)_2^5 = 3 \end{aligned}$$

$$\begin{aligned} f(c) = f_{ave} = 3, \ (x-2)^2 &= 3 \\ c &= 2 \pm \sqrt{3}, \\ c &= 2 + \sqrt{3}. \end{aligned}$$

```
restart;
a:=2; b:=5;
av:=int((x-2)^2,x=a..b)/(b-a);

plot([[t,3,t=0..5],[t,(t-2)^2,t=2..5],
 [5,t,t=0..9],[2,t,t=0..3]],
 scaling=constrained);
```

12. Find 11.51

$$\int t\sin 2t\,dt$$

- **<u>ans:</u>** The idea is to make u' "simpler than" u while v' and v are about the "same".

So it is obviously that $u=t$.

$$u=t, \qquad dv=\sin 2t\,dt$$
$$du=dt, \qquad v=-\frac{1}{2}\cos 2t$$

$$\int udv=uv-\int vdu$$

$$\begin{aligned}\int t\sin 2t\,dt &= -\frac{t}{2}\cos 2t+\frac{1}{2}\int\cos 2tdt\\ &= -\frac{t}{2}\cos 2t+\frac{1}{4}\sin 2t+C\end{aligned}$$

13. Find 11.53

$$\int \tan^{-1}4tdt$$

- **<u>ans:</u>** As there is only one function, we have no choice but let it be u:

$$u=\tan^{-1}4t, \qquad dv=dt$$
$$du=\frac{4}{1+16t^2}, \qquad v=t$$

$$\int udv=uv-\int vdu$$

$$\int\tan^{-1}4t\,dt=t\tan^{-1}4t-\int\frac{4t}{1+16t^2}dt$$

Next, we can change variable:

$$w=1+16t^2,\ dw=32tdt$$

$$\begin{aligned}\int\frac{4t}{1+16t^2}dt &= \int\frac{1}{8}\frac{dw}{w}\\ &= \frac{1}{8}\ln|w|+c=\frac{1}{8}\ln(1+16t^2)+c\end{aligned}$$

$$\int\tan^{-1}4t\,dt=t\tan^{-1}4t-\frac{1}{8}\ln(1+16t^2)+c$$

14. Find 11.55

$$\int_1^2 x^4(\ln x)^2dx$$

- **<u>ans:</u>** The idea is to make u' "simpler than" u while v' and v are about the "same".

But here we have to let $u=(\ln x)^2$ because if this function is part of dv, we would not recover v.

$$u=(\ln x)^2, \qquad dv=x^4\,dx$$
$$du=\frac{2}{x}\ln xdx, \qquad v=\frac{1}{5}x^5$$

$$\int udv=uv-\int vdu$$

$$\begin{aligned}&\int x^4(\ln x)^2\,dt\\ =&\frac{1}{5}x^5(\ln x)^2-\frac{2}{5}\int x^4\ln xdt\end{aligned}$$

Integration by parts again,

$$u=\ln x, \qquad dv=x^4\,dx$$
$$du=\frac{1}{x}dx, \qquad v=\frac{1}{5}x^5$$

$$\begin{aligned}\int x^4\ln xdt &= \frac{1}{5}x^5\ln x-\frac{1}{5}\int x^4\,dx\\ &= \frac{1}{5}x^5\ln x-\frac{1}{25}x^5+c\end{aligned}$$

So

$$\begin{aligned}&\int x^4(\ln x)^2\,dt\\ =&\frac{1}{5}x^5(\ln x)^2-\frac{2}{25}x^5\ln x+\frac{2}{125}x^5+c\end{aligned}$$

$$\begin{aligned}&\int_1^2 x^4(\ln x)^2\,dt\\ =&\left(\frac{1}{5}x^5(\ln x)^2-\frac{2}{25}x^5\ln x+\frac{2}{125}x^5\right)_1^2\\ =&\frac{62}{125}+\frac{32}{5}(\ln 2)^2-\frac{64}{25}\ln 2\end{aligned}$$

15. Find 11.57

$$\int_0^1\frac{y}{e^{2y}}dy$$

- **<u>ans:</u>** The first step is to write the function as a product.

$$\int\frac{y}{e^{2y}}dy=\int ye^{-2y}dy$$

The idea is to make u' "simpler than" u while v' and v are about the "same".

So it is obviously that $u = y$.

$$\begin{aligned} u &= y, & dv &= e^{-2y}\,dy \\ du &= dy, & v &= -\frac{1}{2}e^{-2y} \end{aligned}$$

$$\int udv = uv - \int vdu$$

$$\begin{aligned} \int \frac{y}{e^{2y}}dy &= -\frac{y}{2}e^{-2y} + \frac{1}{2}\int e^{-2y}dy \\ &= -\frac{y}{2}e^{-2y} - \frac{1}{4}e^{-2y} + C \end{aligned}$$

16. Find

11.59

$$\int e^{2y}\sin ydy$$

- **ans:**

The idea is to make u' "simpler than" u while v' and v are about the "same". But here no matter which one is u, the resulted new integral would be the "same" as the original one. So, we would integration by parts twice to get back the original integral, then solve an equation to get the integral.

$$\begin{aligned} u &= \sin y, & dv &= e^{2y}\,dy \\ du &= \cos ydy, & v &= \frac{1}{2}e^{2y} \end{aligned}$$

$$\int udv = uv - \int vdu$$

$$\int e^{2y}\sin ydy = \frac{1}{2}e^{2y}\sin y - \frac{1}{2}\int e^{2y}\cos ydy$$

Repeat

$$\begin{aligned} u &= \cos y, & dv &= e^{2y}\,dy \\ du &= -\sin ydy, & v &= \frac{1}{2}e^{2y} \end{aligned}$$

$$\int e^{2y}\cos ydy = \frac{1}{2}e^{2y}\cos y + \frac{1}{2}\int e^{2y}\sin ydy$$

So

$$\begin{aligned} \int e^{2y}\sin ydy =& \frac{1}{2}e^{2y}\sin y \\ & - \frac{1}{4}e^{2y}\cos y - \frac{1}{4}\int e^{2y}\sin ydy \end{aligned}$$

That is

$$\frac{5}{4}\int e^{2y}\sin ydy = \frac{1}{2}e^{2y}\sin y - \frac{1}{4}e^{2y}\cos y + c$$

$$\int e^{2y}\sin ydy = \frac{2}{5}e^{2y}\sin y - \frac{1}{5}e^{2y}\cos y + c$$