M242 Q10(c) (S. Zhang) (8 points). Name:______________

1. Find the absolute convergence, conditional convergence and divergence by two methods:

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)!}$$

   (a) Root test or ratio test;

   (b) Comparison test, or (extended) limit comaprison test.

   - **ans:** (1) We cannot do it by the root test (in this class). We use the ratio test: +

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{(2n+1)!}{(2n+3)!} \\ &= \frac{1}{(2n+2)(2n+3)} = \frac{1}{\infty} = 0 < 1 \end{aligned}$$

The series converges absolutely by the ratio test.

(2) Comparison test

$$\begin{aligned} a_n &= \frac{1}{(2n+1)!} = \frac{1}{1} \cdot \frac{1}{2} \cdots \frac{1}{2n} \cdot \frac{1}{2n+1} \\ &< \frac{1}{2n} \cdot \frac{1}{2n+1} < \frac{1}{n}\frac{1}{n} = \frac{1}{n^2} \end{aligned}$$

$\sum \frac{1}{n^2}$ converges ($p = 2 > 1$, p-test). By the comparison test, (the smaller one) $\sum \frac{1}{(2n+1)!}$ converges. As it is a positive series, the serives converges absolutely.

If we use the (extended) limit test,

$$\begin{aligned} \lim \frac{a_n}{b_n} &= \lim \frac{\frac{1}{(2n+1)!}}{\frac{1}{n^2}} \\ &= \lim \frac{1}{(2n-1)!} \cdot \frac{1}{2} \cdot \frac{1}{2+\frac{1}{n}} \\ &= 0 \cdot \frac{1}{2} \cdot \frac{1}{2} = 0 < \infty \end{aligned}$$

By the p-test, $\sum \frac{1}{n^2}$ converges. By the (extended) limit test, the (smaller) $\sum \frac{1}{(2n+1)!}$ converges. As it is a positive series, the serives converges absolutely.