M242 Hw10 (S. Zhang) 11.9: 4-7, 11, 15-16, 23-24, 38
11.10: 5-9,14-15,18-19,21-22,30-33, 36-37, 47-48, 51, 55-57, 61-66
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1. (11.9:4) Find the power series representation and its interval of convergence

$$f(x) = \frac{3}{1-x^4}$$

- **ans:**

$$\frac{1}{1-x} = 1 + x + x^2 + ...$$

Pull out 3 and replace x by x^4:

$$\begin{aligned}\frac{3}{1-x^4} =& 3(1 + x^4 + x^8 + ... + x^{4n} + ...)\\ =& 3\sum_{n=0}^{\infty} x^{4n}\\ =& \sum_{n=0}^{\infty} 3x^{4n}\end{aligned}$$

Ratio test:

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|} =& \frac{3x^{4n+4}}{3x^{4n}} = |x|^4 < 1\\ |x| <& 1\end{aligned}$$

Radius of convergence $R = 1$.

Check two end points: $x = -1$ and $x = 1$. Both diverges by the divergence test(or by r-test):

$$\begin{aligned}x = -1,&\ 3 + 3 + 3 + ...\\ x = 1,&\ 3 + 3 + 3 + ...\end{aligned}$$

The interval of convergence:

$$(-1, 1)$$

2. (11.9:16) Find the power series representation and its radius of convergence

$$f(x) = \frac{x^2}{(1-2x)^2}$$

- **ans:** We note that the x^2 on the top does not matter as we can pull it out, or multiply the series by it.

Note that

$$(\frac{1}{1-x})' = \frac{1}{(1-x)^2}$$

we differentiate both sides of

$$\frac{1}{1-x} = 1 + x + x^2 + ...$$

to get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + ... + (n+1)x^n + ...$$

Replace x by $2x$: to get

$$\frac{1}{(1-2x)^2} = 1 + 2(2x) + 3(2x)^2 + ... + (n+1)(2x)^n + ...$$

Multiply the series by x^2

$$\begin{aligned}\frac{x^2}{(1-2x)^2} =& x^2 + 2(2x^3) + ... + (n+1)2^n x^{n+2}\\ =& \sum_{n=0}^{\infty}(n+1)2^n x^{n+2}\end{aligned}$$

Ratio test:

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|} =& \frac{(n+2)2^{n+1}x^{n+3}}{(n+1)2^n x^{n+2}} = \frac{1+\frac{2}{n}}{1+\frac{1}{n}}2|x|\\ &\to 2|x| < 1\\ |x| <& \frac{1}{2}\end{aligned}$$

Radius of convergence $R = 1/2$.

Done.

If we find the interval of convergence, we continue to check two end points: $x = -1/2$

$$\sum_{n=0}^{\infty}(n+1)2^n x^{n+2} = \sum_{n=0}^{\infty}(n+1)(-1)^n\frac{1}{4}$$

The series diverges by the divergence test ($a_n \not\to 0$)

The other end point $x = 1/2$.

$$\sum_{n=0}^{\infty}(n+1)2^n x^{n+2} = \sum_{n=0}^{\infty}(n+1)\frac{1}{4}$$

The seies diverges by the divergence test($a_n \not\to 0$)

The interval of convergence:

$$(-\frac{1}{2}, \frac{1}{2})$$

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1. (11.10:18) Find the Taylor series

$$f(x) = \sin x,\ a = \frac{\pi}{2}$$

- **ans:**

$$\begin{aligned}f(x) &= \sin x, & f(\frac{\pi}{2}) &= 1\\ f'(x) &= \cos x, & f'(\frac{\pi}{2}) &= 0\\ f''(x) &= -\sin x, & f''(\frac{\pi}{2}) &= -1\\ f'''(x) &= -\cos x, & f(a) &= 0\end{aligned}$$

Then the computation repeats.

$$\begin{aligned}\sin x =& f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ...\\ =&1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!}\\ =&-\frac{(x-\frac{\pi}{2})^6}{6!} + ...\end{aligned}$$

Or

$$\sum_{n=0}^{\infty}(-1)^n \frac{(x-\frac{\pi}{2})^{2n}}{(2n)!}$$

2. (11.10:31) Use a well known Maclaurin series to find the Maclaurin series for

$$f(x) = e^x + e^{2x}$$

- **ans:**

$$\begin{aligned}e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\\ e^{2x} &= 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + ...\\ e^x + e^{2x} &= 2 + 3x + \frac{5x^2}{2!}\\ &+ ... + \left(\frac{x^n}{n!} + \frac{(2x)^n}{n!}\right) + ...\\ &= 2 + 3x + \frac{5x^2}{2!}\\ &+ ... + \frac{(1+2^n)x^n}{n!} + ...\end{aligned}$$

Or

$$e^x + e^{2x} = \sum_{n=0}^{\infty} \frac{(1+2^n)x^n}{n!}$$

3. (11.10:57) Use series to evaluate the limit

$$\lim x \to 0 \frac{\sin x - x - \frac{x^3}{6}}{x^5}$$

- **ans:**

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$$

$$\begin{aligned}&\lim_{x\to 0} \frac{\sin x - x - \frac{x^3}{6}}{x^5}\\ &= \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... - x - \frac{x^3}{6}}{x^5}\\ &= \lim_{x\to 0} \frac{\frac{x^5}{5!} - ...}{x^5}\\ &= \lim_{x\to 0} \frac{\frac{1}{5!} - x^2 ...}{1} = \frac{1}{5!}\end{aligned}$$