M242 Hw8 (S. Zhang)
11.3: 6-13, 18-20, 34-37, 42
11.4: 3-6, 11-17, 27-32.

1. (11.3:6) Find the convergence by the integral test.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$$

- **ans:** We find the integral is divergent:

$$\int_1^{\infty} \frac{dn}{\sqrt{n+4}} = \left(\frac{(n+4)^{1/2}}{1/2}\right)_1^{\infty} = \infty - 2\sqrt{5} = \infty$$

The series diverges by the integral test.

2. (11.3:19) Find the convergence by the integral test.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$$

- **ans:** We find the integral is convergent: (integration by parts)

$$\begin{aligned}
&\int_1^{\infty} \frac{\ln n dn}{n^3} \\
\overset{u=\ln n,\ dv=dn/n^3}{\underset{du=dn/n,\ v=-n^{-2}/2}{=}} & -\frac{n^{-2}}{2}\ln n + \int \frac{n^{-2}}{2}\frac{dn}{n} \\
=& -\frac{\ln n}{2n^2} + \frac{1}{2}\int \frac{dn}{n^3} \\
=& -\frac{\ln n}{2n^2} - \frac{1}{4}n^{-2} + c \\
=& \left(-\frac{\ln n}{2n^2} - \frac{1}{4}n^{-2}\right)_1^{\infty} \\
=& (0-0) - (0-\frac{1}{4}) = \frac{1}{4}
\end{aligned}$$

Note that, using L'Hopital rule, we have

$$\lim_{n\to\infty} \frac{\ln n}{n^2} = \lim_{n\to\infty} \frac{1/n}{2n} = \frac{0}{\infty} = 0$$

The series is convergent by the integral test.

3. (11.3:34) Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ correct to 3 decimal places.

- **ans:**

We can either just try to add first few terms to see when to stop to get an asnwer correct to 3 decimal places.

Or (not required – may skip this paragraph) using the integral test to find out the term n needed to get the accuracy, before we add the terms.

$$\begin{aligned}
R_n <& \int_n^{\infty} \frac{1}{x^5}\,dx \\
=& -\frac{1}{4}x^{-4}|_n^{\infty} \\
=& \frac{1}{4n^4}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{4n^4} <& 5\times 10^{-4} \\
n^4 >& 10000/20 \\
n >& 500^{1/4} \sim 4.72
\end{aligned}$$

We need to compute the first 5 terms.

Again, it is enough to try directly as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^5} =& 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + ... \\
=& 1 + 0.03125 + 0.00411522 + 0.0009765625 \\
& + 0.00032 + 0.00012 + ... \\
=& 1.03666 + 0.00012 + ... = 1.037
\end{aligned}$$

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1. (11.4:4) Find the convergence:

$$\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$$

- **ans:** method 1 – comparison test

$$\frac{n^3}{n^4-1} > \frac{n^3}{n^4} = \frac{1}{n}$$

The (smaller) series $\sum 1/n$ diverges (by p-test, $p=1$). By the comparison test, the (big) series $\sum_{n=2}^{\infty} \frac{n^3}{n4-1}$ diverges.

method 2 – limit comparison test

$$\frac{\frac{n^3}{n^4-1}}{\frac{1}{n}} = \frac{1}{1-\frac{1}{n}} \to \frac{1}{1-0} = 1 \neq 0, \infty$$

By the limit comparison test, the two series converge or diverge together. Because $\sum 1/n$ diverges (by p-test, $p=1$), the series $\sum_{n=2}^{\infty} \frac{n^3}{n4-1}$ diverges.

2. (11.4:16) Find the convergence:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$$

- **ans:** method 1 – comparison test

$$\frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$$

The (bigger) series $\sum 1/n^{3/2}$ converges (by p-test, $p = 2 > 1$). By the comparison test, the (small) series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges.

method 2 – limit comparison test

$$\frac{\frac{1}{\sqrt{n^3+1}}}{\frac{1}{\sqrt{n^3}}} = \frac{1}{\sqrt{1+\frac{1}{n^3}}} \to \frac{1}{\sqrt{1+0}} = 1 \neq 0, \infty$$

By the limit comparison test, the two series converge or diverge together. Because $\sum 1/n^{3/2}$ converges (by p-test, $p = 2 > 1$). By the comparison test, the (small) series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges.

3. (11.4:17) Find the convergence:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$$

- **ans:** method 1 – limit comparison test

$$\frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \frac{1}{\sqrt{1+\frac{1}{n^2}}} \to \frac{1}{\sqrt{1+0}} = 1 \neq 0, \infty$$

By the limit comparison test, the two series converge or diverge together. Because $\sum 1/n$ diverges (by p-test, $p = 1$), the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges.

method 2 – comparison test

$$\frac{1}{\sqrt{n^2+1}} > \frac{1}{\sqrt{n^2+n^2}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{n}$$

The (smaller) series $\frac{1}{\sqrt{2}} \sum \frac{1}{n}$ diverges (by p-test, $p = 1$). By the comparison test, the (big) series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges.