

M242 Hw7 (S. Zhang)

11.1: 29, 40, 43-46, 54, 61-64, 67, 68b, 79c, 80b

11.2: 9, 11-12, 22-27, 36-38, 41-43, 65, 72.

1. (11.1:43) Determine whether the sequence converges or diverges. If it converges, find the limit.

$$0, 1, 0, 0, 1, 0, 0, 0, 1, \dots$$

• **ans:** The sequence jumps to 0 and to 1, for ever.

There is no single number such that the terms of the sequence approach.

So the sequence diverges.

2. (11.1:46) Find and show the limit, by the sandwich theorem,

$$a_n = \frac{(-3)^n}{n!}$$

• **ans:** The first few terms:

$$-3, \frac{9}{2}, -\frac{9}{2}, \frac{27}{8}, -\frac{81}{40}, \frac{81}{80}, -\frac{243}{560}, \frac{729}{4480}, \dots$$

We will show the limit first by the sandwich theorem,

$$\lim_{n \rightarrow \infty} \frac{3^n}{n!} = 0$$

$$0 < \frac{3^n}{n!} = \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{3} \cdots \frac{3}{n-1} \cdot \frac{3}{n}$$

$$< \frac{3}{1} \cdot \frac{3}{2} \cdot 1 \cdots 1 \cdot \frac{3}{n}$$

$$= \frac{27/2}{n} \rightarrow 0$$

That is, the both sequences on the left and on the right approach 0, the middle limit is then 0:

$$0 \leftarrow 0 < \frac{3^n}{n!} < \frac{27/2}{n} \rightarrow 0$$

So

$$\lim_{n \rightarrow \infty} \frac{3^n}{n!} = 0$$

Because $\lim |a_n| = 0$ would imply $\lim a_n = 0$ (this is also called the sandwich theorem),

$$\lim_{n \rightarrow \infty} \frac{(-3)^n}{n!} = 0$$

3. (11.1:67) Find limit of the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots$$

• **ans:** Note that

$$a_{n+1} = \sqrt{2a_n}$$

Taking limit on the both sides:

$$L = \sqrt{2L}$$

$$L^2 = 2L$$

$$L^2 - 2L = 0, \quad L = 0, 2.$$

Since a_n is increasing, the extra root $L = 0$ is not the limit. So

$$\lim_{n \rightarrow \infty} a_n = 2$$

1. (11.2:25) Determine convergence of the telescoping series and find the sum if the series converges.

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$$

• **ans:** There are two geometric series here, combined. We separate them, then using the formula

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1+2^n}{3^n} \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \\ &= \frac{1/3}{1-(1/3)} + \frac{2/3}{1-(2/3)} \\ &= \frac{1}{2} + 2 = \frac{5}{2} \end{aligned}$$

2. (11.2:36) Determine convergence of the telescoping series and find the sum if the series converges.

$$\sum_{n=1}^{\infty} \frac{2}{n^2 + 4n + 3}$$

• **ans:** This is a telescoping series. We use the partial fractions to find its sum and show its convergence.

$$\begin{aligned}\frac{2}{n^2 + 4n + 3} &= \frac{2}{(n+1)(n+3)} \\ &= \frac{A}{n+1} + \frac{B}{n+3}\end{aligned}$$

$$A(n+3) + B(n+1) = 2$$

$$n = -1, 2A = 2$$

$$n = -3, -2B = 2$$

$$\frac{2}{n^2 + 4n + 3} = \frac{1}{n+1} - \frac{1}{n+3}$$

$$\begin{aligned}s_n &= \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \dots - \frac{1}{n+1} + \frac{1}{n+3} \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+3} \\ &\rightarrow \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.\end{aligned}$$

3. (11.2:43) Convert the periodic decimals to fractions.

$$3.417417\dots = 3.\overline{417}$$

• **ans:**

$$\begin{aligned}3.\overline{417} &= 3 + 0.417 + 0.000417 + \dots \\ &= 3 + \frac{417}{1000} \left(1 + \frac{1}{1000} + \dots\right) \\ &= 3 + \frac{417}{1000} \frac{1}{1 - (1/1000)} \\ &= 3 + \frac{417}{999} \\ &= \frac{1138}{999}\end{aligned}$$