M242 Hw6 (S. Zhang)
8.1: 1, 7-9, 16-20, 32, 39
11.1: 5-7, 11-15, 19-23, 31-34.

1. (8.1:8) Find arc length:

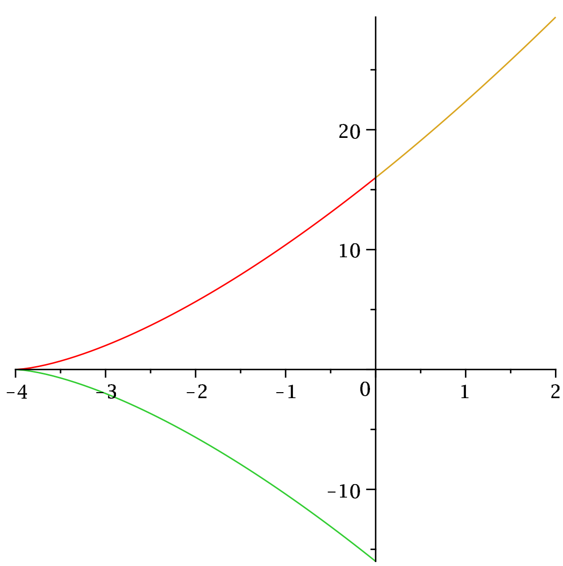
$$y^2 = 4(x+4)^3,\ y > 0,\ 0 \le x \le 2.$$

- **ans:** Taking square root and choose the positive sign:

$$\begin{aligned} y &= 2(x+4)^{3/2} \\ y' &= 3(x+4)^{1/2} \\ \sqrt{1+y'^2} &= \sqrt{1+9(x+4)} \\ &= \sqrt{9x+37} \end{aligned}$$

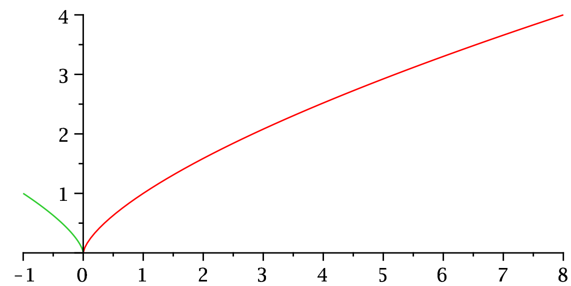
$$\begin{aligned} L &= \int_a^b \sqrt{1+(y')^2}dx \\ &= \int_0^2 \sqrt{9x+37}dx \\ &= \frac{(9x+37)^{3/2}}{(3/2)(9)}\Big|_0^2 \\ &= \frac{2}{37}(55^{3/2} - 37^{3/2}) \\ &= 13.54286690 \end{aligned}$$

From the function graph, the y range goes from about 15 to 30. So the arc length is roughly 15. Our answer is pretty close.



2. (8.1:32) Sketch the curve, find the arc length of the curve from $(0,0)$ to $(1,1)$ by two formulas, and again find the arc length of the curve from $(-1,1)$ to $(8,4)$:

$$y^3 = x^2$$



- **ans:**

$$y = x^{2/3}$$

Method 1, against x.

$$\begin{aligned} y &= x^{2/3} \\ y' &= \frac{2}{3}x^{-1/3} \\ \sqrt{1+(y')^2} &= \sqrt{1+\frac{4}{9}x^{-2/3}} \\ &= \sqrt{\frac{9x^{2/3}+4}{9x^{2/3}}} \\ &= \frac{1}{3}\sqrt{9x^{2/3}+4}x^{-1/3} \end{aligned}$$

$$\begin{aligned} L &= \int_a^b \sqrt{1+(y')^2}dx \\ &= \int_0^1 \frac{1}{3}\sqrt{9x^{2/3}+4}x^{-1/3}dx \end{aligned}$$

$$\begin{aligned} u &= 9x^{2/3}+4 \\ du &= 6x^{-1/3}dx \end{aligned}$$

$$\begin{aligned} \int \sqrt{9x^{2/3}+4}x^{-1/3}dx &= \frac{1}{3}\int \frac{1}{6}\sqrt{u}du \\ &= \frac{1}{18}\frac{2}{3}u^{3/2} \\ &= \frac{1}{27}(9x^{2/3}+4)^{3/2} \end{aligned}$$

$$\begin{aligned} L &= \left(\frac{1}{27}(9x^{2/3}+4)^{3/2}\right)_0^1 \\ &= \frac{1}{27}(13^{3/2} - 4^{3/2}) \\ &= 1.4397 \end{aligned}$$

Note that the curve runs from $(0,0)$ to $(1,1)$. So it is a little longer than the diagonal of the unit square, i.e. a little longer than $\sqrt{2} = 1.414$.

method 2: against y.

$$x = y^{3/2}$$

$$1 + (x')^2 = 1 + \frac{9}{4}y$$

$$\begin{aligned} L &= \int_a^b \sqrt{1+(x')^2}dy \\ &= \int_0^1 \sqrt{1+\frac{9}{4}y}dy \\ &\overset{u=1+\frac{9}{4}y}{=} \int \sqrt{u}\frac{4}{9}du \\ &= \frac{4}{9}(\frac{2}{3}u^{3/2}) \\ &= \left(\frac{8}{27}(1+\frac{9}{4}y)^{3/2}\right)_0^1 \\ &= \frac{8}{27}((13/4)^{3/2}-1) \\ &= 1.4397 \end{aligned}$$

We get the same answer again.

Finally, for the curve from $(-1,1)$ to $(8,4)$, we split it into two parts:

$$-1 \le x \le 0, \ 0 \le x \le 8$$

As the method 2 above is simpler, we use it here too. So the y ranges for the two curves are:

$$0 \le y \le 1, \ 0 \le y \le 4$$

For the first curve

$$x = -y^{3/2}$$

For the second curve

$$x = y^{3/2}$$

For both cases:

$$1 + (x')^2 = 1 + \frac{9}{4}y$$

$$\begin{aligned} L &= \int_0^1 + \int_0^4 \\ &= \int_0^1 \sqrt{1+\frac{9}{4}y}dy + \int_0^4 \sqrt{1+\frac{9}{4}y}dy \\ &= \frac{8}{27}(10\sqrt{10}-1) + \frac{8}{27}(\frac{13\sqrt{13}}{8}-1) \\ &= 10.51312516 \end{aligned}$$

From the graph above, the curve has its horizontal projection from -1 to 8. So the length is at least 9. The above answer looks reasonable.

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1. (11.1:20) Determine whether the sequence converges or diverges. Find the liite if it converges.

$$a_n = \frac{3^{n+2}}{5^n}$$

• **ans:**

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{93^n}{5^n} \\ &= \lim_{n\to\infty} 9\left(\frac{3}{5}\right)^n \\ &= 9(0) = 0 \end{aligned}$$

Note that when $|r| < 1$,

$$\lim_{n\to\infty} r^n = 0$$

2. (11.1:33) Determine whether the sequence converges or diverges. Find the liite if it converges.

$$a_n = n^2 e^{-n}$$

• **ans:** It is of type $\infty \cdot 0$. We need to send one of them to downstairs:

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n^2}{e^n} \\ &\overset{\infty/\infty}{=} \lim_{n\to\infty} \frac{2n}{e^n} \\ &\overset{\infty/\infty}{=} \lim_{n\to\infty} \frac{2}{e^n} \\ &= \frac{2}{\infty} = 0 \end{aligned}$$

3. (11.1:34) Determine whether the sequence converges or diverges. Find the liite if it converges.

$$a_n = n \cos n\pi$$

• **ans:** We can write down the first few terms:

$$\cos\pi, 2\cos 2\pi, 3\cos 3\pi, ...$$

They are

$$-1, 2, -3, 4, -5, 6, ...$$

The sequence jumps toward/between ∞ and $-\infty$. It does not approach a finite number. So the sequence diverges.