M242 Hw4 (S. Zhang) .

1. (7.3:28) Find

$$\int \frac{x^2+1}{(x^2-2x+2)^2}dx$$

• **ans:**

$$x^2-2x+2=(x-1)^2+1$$

It is of type $\sqrt{x^2+a^2}$. Here we can do a change of variable first $u=x-1$. But it is better to expand the idea above, letting

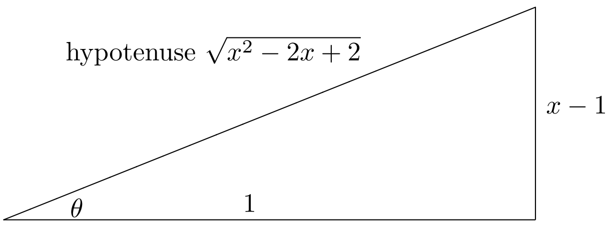
$$x-1=\tan\theta$$

$$dx=\sec^2\theta d\theta$$

$$\begin{aligned} x^2+1 &= (1+\tan\theta)^2+1 \\ &= 1+2\tan\theta+\tan^2\theta+1 \\ &= 2\tan\theta+\sec^2\theta+1 \end{aligned}$$

$$\begin{aligned} &\frac{x^2+1}{(x^2-2x+2)^2}dx \\ &= \int \frac{2\tan\theta+\sec^2\theta+1}{\sec^4\theta}\sec^2\theta d\theta \\ &= \int (2\sin\theta\cos\theta+1+\cos^2\theta)d\theta \\ &= \int (\sin 2\theta+\frac{3}{2}+\frac{1}{2}\cos 2\theta)d\theta \\ &= -\frac{1}{2}\cos 2\theta+\frac{3}{2}\theta+\frac{1}{4}\sin 2\theta+C \\ &= \sin^2\theta+\frac{3}{2}\theta+\frac{1}{2}\sin\theta\cos\theta+C_1 \end{aligned}$$

To find θ and $\cos\theta$, we draw a right triangle:



$$\begin{aligned} &\frac{x^2+1}{(x^2-2x+2)^2}dx \\ &= \frac{(x-1)^2}{x^2-2x+2}+\frac{3}{2}\tan^{-1}(x-1)+\frac{1}{2}\frac{(x-1)}{x^2-2x+2}+C \\ &= \frac{3}{2}\tan^{-1}(x-1)+\frac{1}{2}\frac{(x-3)}{x^2-2x+2}+C \end{aligned}$$

2. (7.3:29) Find

$$\int x\sqrt{1-x^4}dx$$

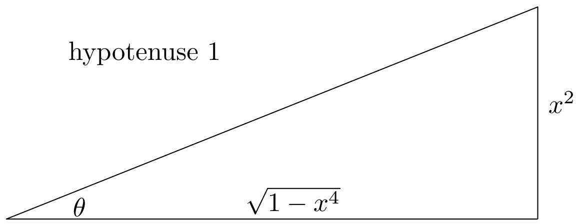
• **ans:** It is of type $\sqrt{a^2-x^2}$. Here we can do a change of variable first $u=x^2$. But it is better to expand the idea above, letting

$$x^2=\sin\theta$$

$$2xdx=\cos\theta d\theta$$

$$\begin{aligned} &\int x\sqrt{1-x^4}dx \\ &= \int \sqrt{1-\sin^2\theta}\frac{1}{2}\cos\theta d\theta \\ &= \int \frac{1}{2}\cos^2\theta d\theta \\ &= \int \frac{1}{4}(1+\cos 2\theta)d\theta \\ &= \frac{\theta}{4}+\frac{1}{8}\sin 2\theta+C \\ &= \frac{\theta}{4}+\frac{1}{4}\sin\theta\cos\theta+C \end{aligned}$$

To find θ and $\cos\theta$, we draw a right triangle:



$$\begin{aligned} &\int x\sqrt{1-x^4}dx \\ &= \frac{1}{4}\sin^{-1}x^2+\frac{1}{4}x^2\sqrt{1-x^4}+C \end{aligned}$$

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1. (7.4:6) Write out the partial fraction forms for $f(x)$. But do not determine the numerical values the coefficients A, B, C etc.

$$(1)\ f(x)=\frac{x^4}{(x^3+x)(x^2-x+3)}$$

$$(2)\ f(x)=\frac{1}{(x^6-x^3)}$$

• **ans:**

(1)

$$(x^3+x)(x^2-x+3)=x(x^2+1)((x-\frac{1}{2})^2+\frac{11}{4})$$

$$\frac{x^4}{(x^3+x)(x^2-x+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2-x+3}$$

(2)

$$x^6 - x^3 = x^3(x^3-1) = x^3(x-1)(x^2+x+1)$$

$$\frac{1}{(x^6-x^3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{Ex+F}{x^2+x+1}$$

2. (7.4:30) Find

$$\int \frac{3x^2+x+4}{x^4+3x^2+2}dx$$

- **ans:**

$$x^4+3x^2+2 = (x^2+1)(x^2+2)$$

$$\frac{3x^2+x+4}{x^4+3x^2+2} = \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+2}$$

$$3x^2+x+4 = (Bx^3+Cx^2+2Bx+2C)+(Dx^3+Ex^2+Dx+E)$$

$$\begin{aligned} B+D &= 0 \\ C+E &= 3 \\ 2B+D &= 1 \\ 2C+E &= 4 \end{aligned}$$

$$B=1,\ C=1,\ D=-1,\ E=2$$

$$\begin{aligned} &\int \frac{3x^2+x+4}{x^4+3x^2+2}dx \\ =&\int \left(\frac{x}{x^2+1} + \frac{1}{x^2+1} - \frac{x}{x^2+2} + \frac{1}{x^2+2}\right)dx \\ =&\frac{1}{2}\ln(x^2+1) + \tan^{-1}x \\ &-\frac{1}{2}\ln(x^2+2) + \sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}} + C \end{aligned}$$

3. (7.4:48) Find

$$\int \frac{\cos x}{\sin^2 x + \sin x}dx$$

- **ans:**

$$u = \sin x,\ du = \cos x dx$$

$$\int \frac{\cos x}{\sin^2 x + \sin x}dx = \int \frac{du}{u^2+u}$$

$$\begin{aligned} \frac{1}{u^2+u} &= \frac{A}{u} + \frac{B}{u+1} \\ &= \frac{1}{u} + \frac{-1}{u+1} \end{aligned}$$

$$\int \frac{\cos x}{\sin^2 x + \sin x}dx = \int \frac{du}{u^2+u}$$

$$\begin{aligned} &= \int \left(\frac{1}{u} + \frac{-1}{u+1}\right)du \\ &= \ln u - \ln|u+1| + c \\ &= \ln|\sin x| + \ln|\sin x + 1| + c \\ &= \ln\frac{|\sin x|}{|\sin x + 1|} + c \end{aligned}$$