Families of optimal order finite elements on rectangle and box grids for the Stokes equations

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Abstract

It is shown that the \(Q_{k+1}-Q_k\) mixed element, approximating the velocity by the continuous \(Q_{k+1}\) polynomials and the pressure by the discontinuous \(Q_k\) polynomials on rectangle and box grids, provides the optimal order of approximation for solving the stationary Stokes equations, for all \(k \geq 2\). A new element, \(Q_{k+1,k} \times Q_{k,k+1}-Q_k\), is proposed and shown as an optimal order method, where different degrees of polynomials in \(x\) and \(y\) are used for different components of the vector velocity. An efficient iterative method is presented which guarantees the iterates converge to the optimal solution and the discrete pressure function is produced as a byproduct of the iteration. Numerical tests are provided confirming the theory.

Keywords. mixed finite element, Stokes, divergence-free element, quadrilateral element, hexahedral element.


1 Introduction

The stability of finite element methods for approximating the incompressible Stokes or Navier-Stokes flows has been a challenge and of great interest in computation. Many techniques and constructions were developed, mostly for low order finite elements, in the past thirty years, cf. [17, 8]. A natural finite element method would be the mixed element which approximates the velocity function by continuous piecewise-polynomials and approximates the pressure function by discontinuous piecewise-polynomials of one degree lower, on triangular, tetrahedral, quadrilateral, or hexahedral grids. Here the method is a truly conforming one in the sense that the incompressibility condition is satisfied pointwise, and that the discrete solution for the velocity is a projection within the divergence-free function space. A fundamental study on the method was done by Scott and Vogelius in 1983 ([18, 19]) that the method is stable and consequently of the optimal order on 2D triangular grids for the \(P_{k+1}-P_k\) element, \(k \geq 3\), provided that the grids have no singular or nearly-singular vertex. Here \(P_k\) stands for the space of polynomials of total degree \(k\) or less. A 2D vertex of a triangulation is singular if all edges meeting at the vertex form two cross lines. It is shown in [18, 19, 27] that the inf-sup condition, or known as Babuška-Brezzi condition, fails as the inf-sup constant degenerates to 0 if any one vertex is becoming singular. The classic theory, which is build on the inf-sup condition, would no longer be applicable, cf. [17, 8]. An analysis was established in [26], which is based on the connection between \(C_1-P_{k+2}\) finite elements and \(C_0-P_{k+1}\) vector finite elements, showing the optimal order

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of convergence of the divergence-free, mixed finite element method. The analysis is extended in [27] to show the Scott-Vogelius $P_{k+1} - P_k$ method works with singular vertices, and extended in [28] to show the method works in 3D on tetrahedral grids for any $k \geq 7$. We note that in the extensions [27] and [28], unlike the cases in [18] and [25], the divergence of the $C_0 - P_{k+1}$ space is a proper subspace of the discontinuous $P_k$ polynomials. The spurious modes in the discrete pressure space are filtered out naturally in an iterative method for the resulting finite element equations, cf. Section 2 below. This avoids the stabilization of $P_{k+1} - P_k$ when singular or near-singular vertices are presented, proposed in [15]. In addition, the discrete pressures are obtained as byproducts of the iteration, greatly simplifying the computation.

In this manuscript, we will show the optimal order of convergence of the $Q_{k+1} - Q_k$ element on rectangular and box grids, where the velocity is approximated by continuous polynomials of separate degree $k+1$ and the pressure is approximated by discontinuous polynomials of separate degree $k$, for all $k \geq 2$ in both 2D and 3D. This is a long-time open problem. Stenberg and Suri showed in [22] the stability, but the sub-optimal order of approximation, for the $Q_{k+1} - Q_{k-1}$ element for all $k \geq 1$ in 2D. Bernardi and Maday showed the stability and the optimal order of convergence for the $Q_{k+1} - P_k$ element, cf. [3]. Given a $C_0 - Q_{k+1}$ space for the velocity, it is preferred to use a maximal discrete space for the pressure, cf. for example, [12, 16, 9]. It is apparently the $Q_{k+1} - Q_k$ element is better than the $Q_{k+1} - Q_{k-1}$ element and the $Q_{k+1} - P_k$ element. In this direction, we propose a new element, approximating the velocity by continuous $Q_{k+1,k} \times Q_{k,k+1}$ polynomials while approximating the pressure by discontinuous $Q_k$ polynomials. Here $Q_{k,l}$ stands for the space of polynomials of degree at most $k$ in $x$ and of degree at most $l$ in $y$. The convergence of the new element is established as a corollary. Because the divergence of a $Q_{k+1,k} \times Q_{k,k+1}$ polynomial is a $Q_k$ polynomial, the new element makes a perfect match for the two discrete spaces. Numerical tests are provided in supporting the theory. We note that the idea of using polynomials of different degrees in different directions for different components is also suggested by Stenberg and Suri in [22], where the $Q_{k+1,k} \times Q_{k,k+1} - Q_{k-1}$ element is studied and shown to have the same approximation results as the $Q_{k+1} - Q_{k-1}$ element. We may extend the results in this manuscript further to the 3D $Q_{k+1,k,k} \times Q_{k,k+1,k} \times Q_{k,k,k+1} - Q_k$ element, and to the cases of quadrilateral and hexahedral grids. We remark that the theories, parallel to our series of work, for the continuous pressure $P_{k+1} - P_k$ elements on triangles, on tetrahedra, and the $Q_{k+1} - Q_k$ elements on quadrilaterals are established in [6], [5] and [1], respectively, extending the Taylor-Hood element [17].

The rest of the paper is organized as follows. In Section 2, we define the finite elements for the Stationary Stokes equations and an iterative method for the resulting linear systems of equations. In Section 3, we show the optimal convergence for the $Q_{k+1} - Q_k$ element, and newly proposed $Q_{k+1,k} \times Q_{k,k+1} - Q_{k-1}$ element. In Section 4, we provide some results of a numerical test.

2 The $Q_{k+1} - Q_k$ and $Q_{k+1,k} \times Q_{k,k+1} - Q_k$ mixed element

In this section, we shall define a class of finite elements for the stationary Stokes equations on rectangular grids. The resulting finite element solutions for the velocity are divergence-free point wise, and of optimal-order approximating the true solution. We will introduce the classic iterated penalty method ([11, 7, 8, 21]) by which the mixed element is reduced to a single divergence-free element. The $Q_{k-1}$ mixed-element solutions for the pressure are obtained as byproducts of the iterated penalty method, approximating the true solution in optimal order as well.
We consider a model stationary Stokes problem: Find the velocity function $\mathbf{u}$ and the pressure $p$ on a 2D or a 3D polygonal domain $\Omega$, which can be subdivided into rectangles or boxes, such that

$$
-\Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega,
$$
$$
\text{div} \mathbf{u} = 0 \quad \text{in } \Omega,
$$
$$
\mathbf{u} = 0 \quad \text{on } \partial \Omega.
$$

The standard variational form for (2.1) is: Find $\mathbf{u} \in H_{1,0}(\Omega)^d$ ($d = 2$ or $3$) and $p \in L_{2,0}(\Omega) := L_2(\Omega)/C = \{ p \in L_2 \mid \int_\Omega p = 0 \}$ such that

$$
a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in H_{1,0}(\Omega)^d, \\
b(\mathbf{u}, q) = 0, \quad \forall q \in L_{2,0}(\Omega).
$$

Here $H_{1,0}(\Omega)^d$ is the subspace of the Sobolev space $H_1(\Omega)^d$ (cf. [10]) with zero boundary trace, and

$$
a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, dx, \\
b(\mathbf{v}, p) = -\int_\Omega \text{div} \, \mathbf{u} \, p \, dx, \\
(\mathbf{f}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx.
$$

We nestedly refine each grid of $\Omega$ by subdividing each rectangle into 4 sub-rectangles (Figure 4) or each box into 8 smaller boxes, connecting some mid-edge points, to obtain a family of grids:

$$
\Omega_h = \{ R \mid R = [x_a, x_b] \times [y_c, y_d] \text{ is a rectangle with the longer edge } h_R, \ h_R \leq h \}, \\
\Omega_h = \{ R \mid R = [x_a, x_b] \times [y_c, y_d] \times [z_e, z_f] \text{ with the longest edge } h_R, \ h_R \leq h \},
$$

for $d = 2$ and $d = 3$, respectively. For simplicity, we restrict our notations to 2D and will make comments for the 3D case.

Let $Q_{k,l}$ be the space of polynomials of degree $k$ or less in $x$ and of degree $l$ or less in $y$:

$$
Q_{k,l} = \left\{ \sum_{i=0}^{k} \sum_{j=0}^{l} q_{ij} x^i y^j \right\}.
$$

As usual, $Q_k = Q_{k,k}$, the space of polynomials of degree $k$ or less in each variable. Then we define the $Q_{k+1}^+Q_k$ and $Q_{k+1,k}^+Q_{k+1}^+$ mixed element spaces by, respectively

$$
\mathbf{V}_h = \{ \mathbf{u}_h \in C(\Omega) \mid \mathbf{u}_h|_R \in Q_{k+1}^+ \forall R \in \Omega_h, \ \text{and} \ \mathbf{u}_h|_{\partial \Omega} = 0 \}, \\
P_h = \{ \text{div} \, \mathbf{u}_h \mid \mathbf{u}_h \in \mathbf{V}_h \}, \\
\tilde{\mathbf{V}}_h = \{ \mathbf{u}_h \in C(\Omega) \mid \mathbf{u}_h|_R \in Q_{k+1,k}^+ \times Q_{k+1}^+ \forall R \in \Omega_h, \ \text{and} \ \mathbf{u}_h|_{\partial \Omega} = 0 \}, \\
\tilde{P}_h = \{ \text{div} \, \mathbf{u}_h \mid \mathbf{u}_h \in \tilde{\mathbf{V}}_h \}.
$$

Since $\int_\Omega P_h = \int_\Omega \text{div} \mathbf{u}_h = \int_{\partial \Omega} \mathbf{u}_h = 0$ for any $p_h \in P_h$, we conclude that

$$
\mathbf{V}_h, \tilde{\mathbf{V}}_h \subset H_{1,0}(\Omega)^2, \quad P_h, \tilde{P}_h \subset L_{2,0}(\Omega),
$$
i.e., the mixed-finite element pairs are conforming. Again, we present the theory for the 2D $Q_{k+1}-Q_k$ first, and we will make comments on the $Q_{k+1,k} \times Q_{k,k+1}-Q_k$ case later to avoid repetition. The resulting system of finite element equations for (2.2) is: Find $u_h \in V_h$ and $p_h \in P_h$ such that

$$
a(u_h, v) + b(v, p_h) = (f, v), \quad \forall v \in V_h, \\
b(u_h, q) = 0, \quad \forall q \in P_h. \tag{2.7}
$$

Traditional mixed-finite elements require the inf-sup condition to guarantee the existence of discrete solutions. As (2.4) provides a compatibility between the discrete velocity and discrete pressure spaces, the linear system of equations (2.7) always has a unique solution, independent of the inf-sup condition.

**Proposition 2.1** ([26]) There is a unique solution in the discrete linear system (2.7) for any polynomial degree, i.e., $k \geq 1$ in (2.3)-(2.4), or in (2.5)-(2.6).

By the second equation in (2.7) and the definition of $P_h$ in (2.4), we conclude that

$$b(u_h, q) = b(u_h, - \text{div} u_h) = \| \text{div} u_h \|_{L_2(\Omega)}^2 = 0 \tag{2.8}$$

and that div $u_h$ is also 0 everywhere. In this case, we call the mixed finite element a divergence-free element. It is apparent that the discrete velocity solution is divergence-free if and only if the discrete pressure finite element space is the divergence of the discrete velocity finite element space, i.e., (2.4). In fact, it is trivial to show ([17, 8, 7, 26]), in the next theorem, that $u_h$ is the unique $a(\cdot, \cdot)$ orthogonal projection from the divergence-free space $Z$ to its subspace $Z_h$, defined by

$$Z := \{ v \in H_{1,0}(\Omega) | \text{div} v = 0 \}, \tag{2.9}$$

$$Z_h := \{ v \in V_h | \text{div} v = 0 \}. \tag{2.10}$$

**Theorem 2.1** The unique solution $u_h$ of (2.7) is divergence-free, and is the $a(\cdot, \cdot)$ orthogonal projection of $u$ of (2.2), i.e.,

$$u_h \in Z_h, \quad a(u - u_h, v) = 0 \quad \forall v \in Z_h.$$

**Proof.** By (2.8), $u_h$ is divergence-free. Letting $v \in Z_h$ in (2.2) and (2.7), we get the above orthogonal projection equation.

We note that by (2.4), $P_h$ is a subspace of discontinuous, piecewise polynomials of separate-degree $k$ or less. As singular vertices are present (see [18, 19]), $P_h$ is a proper subspace. It is difficult to find a local basis for $P_h$. But on the other side, it is the special interest of the method that the space $P_h$ can be omitted in computation and the discrete solutions approximating the pressure function in the Stokes equations can be obtained as byproducts, as we shall see next. We refer to [11, 8, 7, 21] for more information on the following iterative method.

**Definition 2.1** (The iterated penalty method.) Let the initial iterate $u_h^0 = 0$ for the finite element Stokes equation (2.7). The rest iterates $u_h^n$ are defined iteratively to be the unique solution of

$$a(u_h^n, v_h) + \alpha(\text{div} u_h^n, \text{div} v_h) = (f, v_h) + (\text{div} \sum_{j=0}^{n-1} u_h^j, \text{div} v_h) \quad \forall v_h \in V_h,$$
Here \( \alpha \) is positive constant (between 1 and 10 usually.) At the end of iteration, we let

\[
p_h^n = \text{div} \sum_{j=0}^{n} u_j^h.
\]

**Remark 2.1** In the iterated penalty method of Definition 2.1, we need only to do computer coding for the continuous \( Q_{k+1} \) element or \( Q_{k+1,k} \times Q_{k,k+1} \) element for the vector Laplacian like equations.

**Remark 2.2** By Definition 2.1, at the convergence of iteration, \( \text{div} u_h^n = 0 \) and we obtain the solution \( u_h \) of (2.7). Consequently, the unique solution \( p_h \) of (2.7) is obtained as a byproduct.

### 3 Convergence theory

Though the finite equations (2.7) have a unique solution for any \( k \geq 0 \), the approximation cannot be guaranteed, unless \( k \) is large enough. Different from the standard analysis, the convergence of the finite element solutions \((u_h,p_h)\) of (2.7) is not derived from the classic inf-sup condition. Instead, we show the convergence the \( C_0-Q_{k+1} \) element by the approximation property of \( C_1-Q_{k+1} \) piecewise polynomials on rectangles and boxes when solving the biharmonic equations in 2D or 3D.

We relate the Stokes equations (2.2) to the biharmonic equation: Find \( t \in H_{2,0}(\Omega) \) (the closure of \( C_{\infty,0} \) under \( H_2 \) norm), such that

\[
(\Delta t, \Delta s) = -(\text{curl} f, s) \quad \forall s \in H_{2,0}(\Omega).
\]

It is well known [17] that

\[
\text{curl} H_{2,0}(\Omega) = \mathbf{Z} \quad \text{and} \quad \text{curl} t = \mathbf{u}.
\]

Here \( \mathbf{u} \) and \( f \) are defined in (2.2). We will assume the 2D polygonal domain provides a minimum elliptic regularity, i.e., for any \( g \in H_{-1}(\Omega) \), the unique weak solution \( s \in H_{2,0}(\Omega) \) of the biharmonic equation \( \Delta^2 s = g \) satisfies

\[
\|s\|_{H_{r+2}(\Omega)} \leq C\|g\|_{H_{r-2}(\Omega)}, \quad r \geq 1.
\]

For example, Blum and Rannacher showed the \( H_4 \) \((r = 2)\) elliptic regularity in [4] for convex polygons with inner angles smaller than 126.28\(^\circ\). We can find some results on the elliptic regularity in [13].

The corresponding conforming finite element for (3.1) is the space of \( C_1 \) piecewise \( Q_{k+1} \) polynomials on the grid \( \Omega_h \):

\[
S_h = \{ s_h \in C^1(\Omega) \mid s_h|_K \in Q_{k+1}(K) \quad \forall K \in \Omega_h \}.
\]

In the theory to be developed here, we do not need to know the dimension and the construction of the whole space \( S_h \). The optimal order of approximation for a subspace \( S_h \subset S_h \) is needed to guarantee the optimal order of approximation of the divergence-free element. Nevertheless
the full $C_1-Q_{k+1}$ spaces in 2D and 3D are constructed and proved in [29], as follows. Let \( \{\hat{\phi}_i(x)\} \) be the \((k+2)\) basis functions for the \(C_1\) splines of polynomial degree \((k+1)\) on \([0,1]\):

\[
\hat{\phi}'_0(0) = 1, \quad \hat{\phi}'(\frac{j}{k}) = 1, \quad \hat{\phi}'_{k+1}(1) = 1, \quad j = 1, 2, ..., k, \tag{3.5}
\]

and \( \hat{\phi}_i \) is zero when evaluated by the other \((k+1)\) functionals in (3.5). For example, the 4 cubic spline basis functions for \(k = 2\) are

\[
\hat{\phi}_0(x) = x^3 - 2x^2 + x, \quad \hat{\phi}_1(x) = 2x^3 - 3x^2 + 1, \\
\hat{\phi}_2(x) = -2x^3 + 3x^2, \quad \hat{\phi}_3(x) = x^3 - x^2.
\tag{3.6}
\]

The basis functions for \(k = 3\) are

\[
\hat{\phi}_0 = -2x^4 + 5x^3 - 4x^2 + x, \quad \hat{\phi}'_0(0) = 1, \\
\hat{\phi}_1 = -8x^4 + 18x^3 - 11x^2 + 1, \quad \hat{\phi}_1(0) = 1, \\
\hat{\phi}_2 = 16x^4 - 32x^3 + 16x^2, \quad \hat{\phi}_2(\frac{1}{2}) = 1, \\
\hat{\phi}_3 = -8x^4 + 14x^3 - 5x^2, \quad \hat{\phi}_3(1) = 1, \\
\hat{\phi}_4 = 2x^4 - 3x^3 + x^2, \quad \hat{\phi}_4(1) = 1.
\tag{3.7}
\]

The basis functions in 2D and 3D are constructed by the tensor products of (3.5), depicted in Figures 1 and 2. Then they are mapped to individual elements, \([x_m, x_{m+1}] \times [y_n, y_{n+1}]\), by

\[
\phi_1(x) = \begin{cases} 
\hat{\phi}_i(\frac{x-x_m}{h}), & 0 < i < k + 1, \\
\hat{h}\phi_i(\frac{x-x_m}{h}), & i = 0, k + 1,
\end{cases}, \\
\phi_j(y) = \begin{cases} 
\hat{\phi}_j(\frac{y-y_n}{h}), & 0 < j < k + 1, \\
\hat{h}\phi_j(\frac{y-y_n}{h}), & j = 0, k + 1.
\end{cases} \tag{3.8}
\]

It is shown that $C_1-Q_{k+1}$ space \(S_h\) in (3.4) is precisely the span of local basis functions defined in (3.5–3.8). By the nodal interpolation operator, it is standard to get, cf. [10], that

\[
\inf_{t_h \in S_h} \|s - t_h\|_{H^2} \leq C H_{\min(k,r)}^{\min(k,r)} \|s\|_{H_{r+2}(\Omega)} \quad \forall s \in H_{2,0} \cap H_{r+2}, \quad r \geq 1. \tag{3.9}
\]

We need the following result on the regular inversion of the divergence operator: The solution \(U \in H_{1,0}(\Omega)^2\) of

\[
\text{div} \, U = F \tag{3.10}
\]

satisfies

\[
\|U\|_{H_r(\Omega)^2} \leq C \|F\|_{H_{r-1}(\Omega)} \quad \text{for some } r \geq 1, \tag{3.11}
\]
that of [26]. But for completeness, we present here the main steps of proof again.

Theorem 3.1 Let the smooth solution $t$ in (3.1) have the elliptic regularity (3.3). Let the smooth solution $U$ in (3.10) have the bounded regular inversion (3.11). The unique solutions $(u_h, p_h)$, in $V_h \times P_h$ or in $\tilde{V}_h \times \tilde{P}_h$, of the discrete Stokes equations (2.7) approximate that of (2.2) in the optimal order:

$$
\|u - u_h\|_{H^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq Ch^{\min\{k+1,r\}}\|f\|_{H^{r-1}(\Omega)^2}, \quad r \geq 1.
$$

(3.12)

Proof. Let $t_h \in S_h$ be the finite element solution for the biharmonic problem (3.1), i.e.,

$$(\Delta t_h, \Delta s_h) = -(\text{curl} f, s_h) \quad \forall s_h \in S_h.
$$

(3.13)

Subtracting (3.13) from (3.1), by Cea’s lemma [7, 10], (3.9) and (3.3),

$$
|t - t_h|_{H^2(\Omega)} \leq Ch^{\min\{k+1,r\}}\|\text{curl} f\|_{H^{r-2}(\Omega)} \leq Ch^{\min\{k+1,r\}}\|f\|_{H^{r-1}(\Omega)^2}.
$$

(3.14)

By Theorem 2.1 and (3.2), since $\text{curl} t_h \in Z_h$, it follows that

$$
\|u - u_h\|_{H^1(\Omega)^2} = \inf_{v \in Z_h} |u - v|_{H^1(\Omega)^2} \leq \inf_{s_h \in S_h} |\text{curl} t - \text{curl} s_h|_{H^1(\Omega)^2}
\leq |\text{curl} t - \text{curl} t_h|_{H^1(\Omega)^2} \leq C|t - t_h|_{H^2(\Omega)}
\leq Ch^{\min\{k+1,r\}}\|f\|_{H^{r-1}(\Omega)^2}.
$$

(3.15)

The $L^2$ error of the velocity is bounded by the $H^1$ error, ensured by the Poincaré inequality. Note that in (3.12), we used the inclusion relations:

$$
\text{curl} S_h \subset V_h \quad \text{and} \quad \text{curl} S_h \subset \tilde{V}_h.
$$

By (2.2) and (2.7), we have

$$
(p - p_h, \text{div} v) = (\nabla (u - u_h), \nabla v) \quad \forall v \in V_h \quad \text{or} \quad \tilde{V}_h.
$$

(3.16)

Therefore we introduce $\tilde{p} \in L_{2,0}(\Omega)$ such that

$$
(\tilde{p}, \text{div} v) = (\nabla (u - u_h), \nabla v) \quad \forall v \in H_{1,0}(\Omega)^2.
$$

(3.17)
Because \( \text{div} \mathbf{u} = \text{div} \mathbf{u}_h = 0 \), the existence and the uniqueness of \( \tilde{p} \) in (3.17) are guaranteed by the inf-sup condition for the smooth functions ([17, 8]). Letting \( F = \tilde{p} \) in (3.10), we have an \( \mathbf{U} \in H_{1,0}(\Omega)^2 \cap H_r(\Omega)^2 \) such that \( \text{div} \mathbf{U} = \tilde{p} \) and

\[
\| \mathbf{U} \|_{H_1(\Omega)^2} \leq C \| \tilde{p} \|_{L_2(\Omega)}.
\]

Let \( \mathbf{v} = \mathbf{U} \) in (3.17).

\[
\| \tilde{p} \|_{L_2(\Omega)}^2 \leq |\mathbf{u} - \mathbf{u}_h|_{H_1} |\mathbf{U}|_{H_1} \leq C |\mathbf{u} - \mathbf{u}_h|_{H_1} \| \tilde{p} \|_{L_2(\Omega)}.
\]

(3.18)

By (3.16) and (3.17), because \( q_h = \text{div} \mathbf{v}_h \) for some \( \mathbf{v}_h \in \mathbf{V}_h \), we get that

\[
(p - p_h - \tilde{p}, q_h) = 0 \quad \forall q_h \in P_h.
\]

Hence,

\[
\| p - p_h - \tilde{p} \|_{L_2(\Omega)}^2 = (p - p_h - \tilde{p}, p - q_h - \tilde{p}) \leq \| p - p_h - \tilde{p} \|_{L_2(\Omega)} \left( \| p - q_h \|_{L_2(\Omega)} + \| \tilde{p} \|_{L_2(\Omega)} \right),
\]

(3.19)

where \( q_h \in P_h \) is arbitrary. Finally, letting \( F = p \) in (3.10), there is a \( \mathbf{U} \in H_{1,0}(\Omega)^2 \cap H_r(\Omega)^2 \) such that \( \text{div} \mathbf{U} = p \) and

\[
\| \mathbf{U} \|_{H_1(\Omega)^2} \leq C \| p \|_{L_2(\Omega)}.
\]

(3.20)

For the smooth function \( U \) in (3.20), we let \( \mathbf{v}_h \in \mathbf{V}_h \) approximate \( \mathbf{U} \) in optimal order, for example, the edge-averaging interpolation of \( \mathbf{U} \) defined in [20]. Let \( q_h = \text{div} \mathbf{v}_h \). We conclude that

\[
\| p - q_h \|_{L_2(\Omega)} = \| \text{div}(\mathbf{U} - \mathbf{v}_h) \|_{L_2(\Omega)} \leq \| \mathbf{U} - \mathbf{v}_h \|_{H_1(\Omega)^2}
\]

\[
\leq C h^{\min\{k+1,r\}} \| \mathbf{U} \|_{H_{r+1}(\Omega)^2} \leq C h^{\min\{k+1,r\}} \| \mathbf{p} \|_{H_r(\Omega)^2} \leq C h^{\min\{k+1,r\}} \| \mathbf{f} \|_{H_{r-1}(\Omega)^2},
\]

(3.21)

where the standard elliptic regularity on \( p \) for the Stokes equations (2.1) is applied in the last step. Combining (3.18), (3.19) and (3.21), (3.12) is shown:

\[
\| p - p_h \|_{L_2(\Omega)} \leq \| p - p_h - \tilde{p} \|_{L_2(\Omega)} + \| \tilde{p} \|_{L_2(\Omega)} \leq \| p - q_h \|_{L_2(\Omega)} + 2 \| \tilde{p} \|_{L_2(\Omega)} \leq C h^{\min\{k+1,r\}} \| \mathbf{f} \|_{H_{r-1}(\Omega)^2} + 2 C |\mathbf{u} - \mathbf{u}_h|_{H_1(\Omega)^2}
\]

\[
\leq C h^{\min\{k+1,r\}} \| \mathbf{f} \|_{H_{r-1}(\Omega)^2}.
\]

**Corollary 3.1** Assume that the elliptic regularity (3.3) the bounded regular inversion (3.11) hold for \( \Omega \) in 3D. The unique solution \((\mathbf{u}_h, p_h)\) of the 3D \( Q_k+1-Q_k \) element method of the discrete Stokes equations (2.7) approximate that of (2.2) in the optimal order:

\[
\| \mathbf{u} - \mathbf{u}_h \|_{H_1(\Omega)^2} + \| p - p_h \|_{L_2(\Omega)} \leq C h^{\min\{k+1,r\}} \| \mathbf{f} \|_{H_{r-1}(\Omega)^2}, \quad r \geq 1.
\]

(3.22)

**Proof.** The analysis remains the same except the biharmonic equation (3.1) is now a vector equation. In 3D, the potential function \( \mathbf{t} \) is not unique for each \( \mathbf{u} \), unlike the 2D situation. However, by requiring \( \text{div} \mathbf{t} = 0 \), we can still have the uniqueness. For details, please read [28].
4 Numerical tests

In this section, we report some results of numerical experiments of the \( Q_{k+1,k} \times Q_{k,k+1} \)-\( Q_k \) element and the \( Q_{k+1} - Q_k \) element for the stationary Stokes equations (2.1) on the unit square. We choose a simple exact solution so that the right hand side function \( f \) for (2.1) is

\[
f = -\Delta \mathbf{curl} g - \nabla g_{xx} = \begin{pmatrix} -g_{yxx} - g_{yyy} - g_{xxx} \\ g_{xxx} + g_{xxy} - g_{yxx} \end{pmatrix},
\]

where

\[
g = 2^8 (x - x^2)^2 (y - y^2)^2.
\]

The continuous solution for the Stokes equations (2.1) is (see Figure 3)

\[
\mathbf{u} = \mathbf{curl} g, \quad p = -g_{xx}.
\]

The grids \( \Omega_h \) are depicted in Figure 4, i.e., each squares are refined into 4 sub-squares each level. In Table 1 we list various norms and orders of convergence of errors for the finite element solutions in the spaces \( \mathbf{V}_h \times P_h \) defined in (2.5)-(2.6), i.e. for the \( Q_{k+1,k} \times Q_{k,k+1} - Q_k \) element. Here we do enough iterated penalty iterations defined in Definition 2.1 until the iterative error is smaller than the truncation error each time. The \( H_1 \) and \( L_2 \) errors and convergence orders, reported in Table 1, are consistent with the error bound proved in Theorem 3.1. In Table 1, the nodal errors are of the optimal order too, but not proved theoretically in this paper. To be precise, the order of convergence for the velocity when using the \( Q_{3,2} \times Q_{2,3} - Q_2 \) element, or using the \( Q_{4,3} \times Q_{3,4} - Q_3 \) element is one order, or two orders higher than that predicted by the theory, respectively. This might be due to the superconvergence of finite elements, cf. [24]. Or it might be caused by, in addition, the special solution (4.2) used in the computation, noting that the convergence orders for the pressure are not higher. We plot the error for the second
Table 1: The errors by spaces $\tilde{V}_h \times \tilde{P}_h$ defined in (2.5)-(2.6), on Figure 4 grids.

| level | $|u - u_h|_{H^1}$ $h^m$ | $|u - u_h|_{L^\infty}$ $h^n$ | $\|p - p_h\|_{L^2}$ $h^m$ | $\|p - p_h\|_{L^\infty}$ $h^m$ |
|-------|-----------------|------------------|-----------------|-----------------|
| 2     | 1.00989         | 0.1231114        | 11.024          | 2.82776         |
| 3     | 0.10747         | 3.2              | 2.755           | 2.0             |
| 4     | 0.01274         | 3.1              | 0.671           | 2.0             |
| 5     | 0.00157         | 3.0              | 0.170           | 2.0             |
| 6     | 0.00019         | 3.0              | 0.043           | 2.0             |
| 7     | 0.00002         | 3.0              | 0.011           | 2.0             |

For the $Q_{3,2} \times Q_{2,3}-Q_2$ element

| level | $|u - u_h|_{H^1}$ $h^m$ | $|u - u_h|_{L^\infty}$ $h^n$ | $\|p - p_h\|_{L^2}$ $h^m$ | $\|p - p_h\|_{L^\infty}$ $h^m$ |
|-------|-----------------|------------------|-----------------|-----------------|
| 2     | 0.014787998     | 0.0040368767     | 0.775887        | 2.0             |
| 3     | 0.000755147     | 4.3              | 0.0113266       | 2.8             |
| 4     | 0.000028532     | 4.7              | 0.0015266       | 2.9             |
| 5     | 0.000009666     | 4.9              | 0.0001954       | 3.0             |
| 6     | 0.000000031     | 4.9              | 0.0000245       | 3.0             |
| 7     | 0.000000001     | 5.0              | 0.0000030       | 3.0             |

For the $Q_{4,3} \times Q_{3,4}-Q_3$ element

In Table 2, we list the errors and convergence orders from the numerical results when using the $Q_{2,1} \times Q_{1,2}-Q_1$ element. A little surprising, the results are of optimal orders too, but they are not covered by our theory. Further studies are needed to understand and explain the results in Table 2.

| level | $|u - u_h|_{H^1}$ $h^m$ | $|u - u_h|_{L^\infty}$ $h^n$ | $\|p - p_h\|_{L^2}$ $h^m$ | $\|p - p_h\|_{L^\infty}$ $h^m$ |
|-------|-----------------|------------------|-----------------|-----------------|
| 2     | 0.00000106532   | 5.3              | 0.0000010262    | 5.9             |
| 3     | 0.00000000029   | 11.8             | 0.00000000002   | 12.2            |
| 4     | 0.00000000000   | 13.8             | 0.00000000000   | 13.8            |

In Table 3, we list the errors and convergence orders from the numerical results when using the $Q_{2,1} \times Q_{1,2}-Q_1$ element. A little surprising, the results are of optimal orders too, but they are not covered by our theory. Further studies are needed to understand and explain the results in Table 3.

Finally, in Table 4, we report numerical results for the $Q_{k+1}-Q_k$ element. Surprisingly, this method requires much more calculation but produces much worse results than those computed by the $Q_{k+1}-Q_{k+1}$ element. It is interesting to observe that the $Q_2-Q_1$ element diverges. For the $Q_3-Q_2$ element, the convergence of the velocity is a little better than that predicted by the theory, but it barely reaches the convergence order of the pressure. For the $Q_4-Q_3$ element, since the exact velocity and the exact pressure are in the corresponding finite element spaces, we are supposed to get the exact numerical solutions if there is no round-off error nor
Table 3: The errors by the $Q_{2,1} \times Q_{1,2}$-$Q_1$ element, on Figure 4 grids.

| level | $|u - u_h|_{H^1} h^n$ | $|u - u_h|_{L_\infty} h^n$ | $\|p - p_h\|_{L_2} h^m$ | $\|p - p_h\|_{L_\infty} h^m$ |
|-------|-----------------------|-----------------------|-----------------|------------------|
| 3     | 1.2110203             | 0.1994708             | 25.9107521      | 6.6670602        |
| 4     | 0.3119692 2.0         | 0.0457680 2.1         | 10.5018161 1.3  | 1.4463214 2.2    |
| 5     | 0.0783616 2.0         | 0.0110037 2.1         | 4.9049518 1.1   | 0.3727838 2.0    |
| 6     | 0.0196081 2.0         | 0.0027090 2.0         | 2.4052678 1.0   | 0.0927318 2.0    |
| 7     | 0.0049030 2.0         | 0.0006762 2.0         | 1.1964874 1.0   | 0.0234598 2.0    |

quadrature error. But it seems the results are not as good as one would expect. The reason is that the iterated penalty method does not converge fast and it accumulates numerical errors.

Table 4: The errors by spaces $V_h \times P_h$ defined in (2.3)-(2.4), on Figure 4 grids.

| level | $|u - u_h|_{H^1} h^n$ | $|u - u_h|_{L_\infty} h^n$ | $\|p - p_h\|_{L_2} h^m$ | $\|p - p_h\|_{L_\infty} h^m$ |
|-------|-----------------------|-----------------------|-----------------|------------------|
|       | For the $Q_2$-$Q_1$ element |
| 2     | 13.56465              | 2.99999               | 665.74          | 185.699          |
| 3     | 7.24861 0.9           | 1.03747 1.5           | 1411.47 -1.1    | 292.334 -0.7     |
| 4     | 3.55098 1.0           | 0.29862 1.8           | 2684.15 -0.9    | 335.117 -0.2     |
| 5     | 1.57715 1.2           | 0.07365 2.0           | 4713.06 -0.8    | 339.128 -0.1     |
|       | For the $Q_3$-$Q_2$ element |
| 2     | 3.4935 2.1            | 0.3963 2.2            | 376.21 0.1      | 87.968 0.7       |
| 3     | 0.7984 2.1            | 0.0430 3.2            | 355.91 0.1      | 46.189 0.9       |
| 4     | 0.1697 2.2            | 0.0043 3.3            | 311.59 0.2      | 23.212 1.0       |
| 5     | 0.0303 2.5            | 0.0003 3.5            | 225.13 0.5      | 8.447 1.5        |
|       | For the $Q_4$-$Q_3$ element |
| 2     | 0.0000000007          | 0.0000000005          | 0.0001010      | 0.000002         |
| 3     | 0.0000000084          | 0.00000000164         | 0.0003452      | 0.000044         |
| 4     | 0.000000107           | 0.0000000039          | 0.0016921      | 0.000152         |
| 5     | 0.000000023           | 0.000000004           | 0.0014434      | 0.000074         |

References


Figure 5: The error of the second component of $u$ on the level 3 grid ($Q_{3,2} \times Q_{2,3}-Q_2$).


Figure 6: The error of $p$ on the level 3 grid ($Q_{3,2} \times Q_{2,3}-Q_2$).


