A family of $Q_{k+1,k} \times Q_{k,k+1}$ divergence-free finite elements on rectangular grids

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Abstract

It is shown that the conforming $Q_{k+1,k} \times Q_{k,k+1}$-$Q_k^-$ mixed element is stable, and provides optimal order of approximation for the Stokes equations, on rectangular grids, for all $k \geq 2$. Here $Q_{k+1,k}$ stands for the space of continuous piecewise-polynomials of degree $(k+1)$ or less in x direction and of degree k or less in y direction. Q_k^- is the space of discontinuous polynomials of separated degree k or less, with spurious modes filtered. To be precise, Q_k^- is the divergence of the discrete velocity space $Q_{k+1,k} \times Q_{k,k+1}$. Polynomials of different degrees in x and y are used for different components of velocity so that the resulting finite element solution is also divergence free point wise. This is the first divergence-free element found on non triangular grids. An efficient iterative method is presented where the discrete pressure function is produced as a byproduct. Numerical tests are provided confirming the theory.

Keywords. mixed finite element, Stokes, divergence-free element, quadrilateral element, rectangular grids.

AMS subject classifications (2000). 65M60, 65N30, 76M10, 76D07.

1 Introduction

It is a challenge, and of great interest in computation too, to construct stable pairs of finite elements in computing the incompressible Stokes or Navier-Stokes flows. Many techniques were developed, mostly for low order finite elements, in the past thirty years, cf. [14, 6]. A natural finite element method would be the mixed element which approximates the velocity function by continuous piecewise-polynomials and approximates the pressure function by discontinuous piecewise-polynomials of one degree less. Here the method can be truly conforming in the sense that the incompressible condition would be satisfied point wise, and that the discrete solution for the velocity is a projection within the divergence-free function space. A fundamental study on the method was done by Scott and Vogelius in 1984 ([15, 16]) that the method is stable and consequently of the optimal order on 2D triangular grids for the P_{k+1}-P_k element, $k \geq 3$, provided that the grids have no nearly-singular vertex. Here P_k stands for the space of polynomials of total degree k or less.

In this manuscript, we will show the stability and the optimal order of convergence of a $Q_{k+1,k} \times Q_{k,k+1}$-$Q_k^-$ element, where the velocity is approximated by continuous polynomials, and the pressure is approximated by discontinuous polynomials of separate degree k with

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spurious modes filtered, for all $k \geq 2$, on rectangular grids. Here the polynomial sets are defined by

$$Q_{m,k} = \left\{ \sum_{i=0}^{m} \sum_{j=1}^{k} v_{ij} x^i y^j \right\}, \quad \text{and } Q_k = Q_{k,k}. \tag{1.1}$$

This work is related to a long-time research problem, the stability of Q_{k+1}-Q_k mixed finite element. Stenberg and Suri showed in [19] the stability, but a sub-optimal order of approximation, for the Q_{k+1}-Q_{k-1} element for all $k \geq 1$ in 2D. Bernardi and Maday showed the stability and the optimal order of convergence for the Q_{k+1}-P_k element, cf. [4]. Ainsworth and Coggins proved the stability and the optimal order of convergence for the Taylor-Hood type Q_{k+1}-Q_k element, where the pressure space is continuous too, cf. [1]. Brezzi and Falk showed that the Q_{k+1}-Q_k element is unstable in [7], for any $k \geq 0$, in the traditional sense.

Given the C^0-Q_{k+1} space for the velocity, it is preferable (cf. [11, 13, 8]) to use a bigger discrete space for the pressure such that the incompressible condition can be well preserved. The maximal possible (in space dimension) discrete pressure space is the divergence of the discrete velocity space. If so, the discrete solution for the velocity would be also divergence free. Nevertheless, we do not know yet if such a pair of mixed finite element would be stable or not, and how to amend it if it is unstable, cf. [6, 7]. We note that $\operatorname{div}(Q_{k+1}) \not\subset Q_k$ and $\operatorname{div}(Q_{k+1}) \not\supset Q_k$, in this case, while Brezzi and Falk proved instability of the pair Q_{k+1}-Q_k. In the opposite direction, to find a divergence-free element on rectangular grids, we seek a finite element space whose divergence would be on to the discontinuous Q_k space. This leads to the new element, approximating the velocity by continuous $Q_{k+1,k} \times Q_{k,k+1}$ polynomials while approximating the pressure by discontinuous polynomials $Q_k^- = \operatorname{div}(Q_{k+1,k} \times Q_{k,k+1})$ (see (2.4) below for the precise definition where $Q_k^- = P_h$). We note that, by choosing $\operatorname{div}(Q_{k+1,k} \times Q_{k,k+1})$ as the discrete finite element space for the pressure, the spurious modes in discontinuous Q_k space are filtered out automatically. On the other side, the discrete velocity is divergence-free if and only if the discrete pressure space is the divergence of the discrete velocity space. Because the divergence of a $Q_{k+1,k} \times Q_{k,k+1}$ polynomial is a Q_k polynomial, the new element makes a perfect match for the two discrete spaces and produces divergence free solutions for the velocity. We will establish the stability and the optimal-order convergence for the new element for all $k \geq 2$. Numerical tests are provided in supporting the theory.

We note that the idea of using polynomials of different degrees in different directions for different components is not new, for example, suggested by Stenberg and Suri in [19], where the $Q_{k+1,k} \times Q_{k,k+1}$-$Q_{k-1}$ element is studied and shown to have the same approximation results as the Q_{k+1}-Q_{k-1} element. Our new element increases the convergence order by one while having the same order of computation. In addition, the discrete solution for the velocity is divergence free. This is the first divergence-free element found on non triangular grids. We note that our results can be extended to a 3D divergence-free element, $Q_{k+1,k,k} \times Q_{k,k+1,k} \times Q_{k,k,k+1}$-$Q_k^-$, $k \geq 2$. Another interesting question is whether the new element remains stable for $k = 1$, see the numerical results and remarks in the last section.

The rest of the paper is organized as follows. In Section 2, we define the finite element for the Stationary Stokes equations and an iterative method for the resulting linear systems of equations. In Section 3, we show the optimal convergence for the newly proposed $Q_{k+1,k} \times Q_{k,k+1}$-$Q_k^-$ element. In Section 4, we provide some results of a numerical test.

2 The $Q_{k+1,k} \times Q_{k,k+1}$-$Q_k^-$ mixed element

In this section, we shall define a class of finite elements for the stationary Stokes equations on rectangular grids. The resulting finite element solutions for the velocity are divergence-free point wise. We will introduce the classic iterated penalty method ([10, 5, 6, 18]) by which the mixed element is reduced to a single divergence-free element. The mixed-element solutions for the pressure are obtained as byproducts in the method.

We consider a model stationary Stokes problem: Find the velocity function $\mathbf{u}$ and the pressure p on a 2D polygonal domain Ω, which can be subdivided into rectangles , such that

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

The standard variational form for (2.1) is: Find $\mathbf{u} \in H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega) := L^2(\Omega)/C = \{p \in L^2 \mid \int_\Omega p = 0\}$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) && \forall \mathbf{v} \in H_0^1(\Omega)^2, \\ b(\mathbf{u}, q) &= 0 && \forall q \in L_0^2(\Omega). \end{aligned} \tag{2.2}$$

Here $H_0^1(\Omega)^2$ is the subspace of the Sobolev space $H^1(\Omega)^2$ (cf. [9]) with zero boundary trace, and

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, dx, \\ b(\mathbf{v}, p) &= -\int_\Omega \operatorname{div} \mathbf{v} \, p \, dx, \\ (\mathbf{f}, \mathbf{v}) &= \int_\Omega \mathbf{f} \, \mathbf{v} \, dx. \end{aligned}$$

We nestedly refine each grid of Ω by subdividing each rectangle into 4 sub-rectangles, cf. Figure 5:

$$\Omega_h = \left\{ K \mid \cup K = \overline{\Omega}, \ K = [x_a, x_b] \times [y_c, y_d] \ \text{ with size } \ h_K = \max\{x_b - x_a, y_d - y_c\} \le h \right\}.$$

Let the polynomial spaces $Q_{k,l}$ and Q_k be defined in (1.1). The $Q_{k+1,k} \times Q_{k,k+1}$-$Q_k^-$ mixed element spaces are

$$\mathbf{V}_h = \left\{ \mathbf{u}_h \in C(\Omega)^2 \mid \mathbf{u}_h|_K \in Q_{k+1,k} \times Q_{k,k+1} \ \forall K \in \Omega_h, \text{ and } \ \mathbf{u}_h|_{\partial\Omega} = 0 \right\}, \tag{2.3}$$

$$P_h = \{\operatorname{div} \mathbf{u}_h \mid \mathbf{u}_h \in \mathbf{V}_h\}. \tag{2.4}$$

Since $\int_\Omega p_h = \int_\Omega \operatorname{div} \mathbf{u}_h = \int_{\partial\Omega} \mathbf{u}_h = 0$ for any $p_h \in P_h$, we conclude that

$$\mathbf{V}_h \subset H_0^1(\Omega)^2, \quad P_h \subset L_0^2(\Omega),$$

i.e., the mixed-finite element pair is conforming. The resulting system of finite element equations for (2.2) is: Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in P_h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) &= (\mathbf{f}, \mathbf{v}) && \forall \mathbf{v} \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q) &= 0 && \forall q \in P_h. \end{aligned} \tag{2.5}$$

Traditional mixed-finite elements require the inf-sup condition to guarantee the existence of discrete solutions. As (2.4) provides a compatibility between the discrete velocity and discrete pressure spaces, the linear system of equations (2.5) always has a unique solution, independent of the inf-sup condition.

Proposition 2.1 *([21]) There is a unique solution in the discrete linear system (2.5) for any polynomial degree $k \geq 1$ in (2.3)-(2.4).* ∎

By the second equation in (2.5) and the definition of P_h in (2.4), we conclude that

$$b(\mathbf{u}_h, q) = b(\mathbf{u}_h, -\operatorname{div}\mathbf{u}_h) = \|\operatorname{div}\mathbf{u}_h\|^2_{L^2(\Omega)^2} = 0 \tag{2.6}$$

and that $\operatorname{div}\mathbf{u}_h$ is also point wise 0. In this case, we call the mixed finite element a divergence-free element. It is apparent that the discrete velocity solution is divergence-free if and only if the discrete pressure finite element space is the divergence of the discrete velocity finite element space, i.e., (2.4). In fact, it is trivial to show ([14, 6, 5, 21]), in the next theorem, that $\mathbf{u}_h$ is the unique $a(\cdot,\cdot)$ orthogonal projection from the divergence-free space $\mathbf{Z}$ to its subspace $\mathbf{Z}_h$, defined by

$$\mathbf{Z} := \left\{\mathbf{v} \in H^1_0(\Omega)^2 \mid \operatorname{div}\mathbf{v} = 0\right\}, \tag{2.7}$$

$$\mathbf{Z}_h := \{\mathbf{v} \in \mathbf{V}_h \mid \operatorname{div}\mathbf{v} = 0\}. \tag{2.8}$$

Theorem 2.1 *The unique solution $\mathbf{u}_h$ of (2.5) is divergence-free, and is the $a(\cdot,\cdot)$ orthogonal projection of $\mathbf{u}$ in (2.2), i.e.,*

$$\mathbf{u}_h \in \mathbf{Z}_h, \quad a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{Z}_h.$$

Proof. By (2.6), $\mathbf{u}_h$ is divergence-free. Letting $\mathbf{v} \in \mathbf{Z}_h$ in (2.2) and (2.5), we get the above orthogonal projection equation. ∎

We note that by (2.4), P_h is a subspace of discontinuous, piecewise polynomials of separate-degree k or less. As singular vertices are present (see [15, 16]), P_h is a proper subset of the discontinuous piecewise Q_k polynomials. It is difficult to find a local basis for P_h. But on the other side, it is the special interest of the method that the space P_h can be omitted in computation and the discrete solutions approximating the pressure function in the Stokes equations can be obtained as byproducts, as we shall see next. We refer to [10, 6, 5, 18] for more information on the following iterative method.

Definition 2.1 (The iterated penalty method.) Let the initial iterate $\mathbf{u}_h^0 = \mathbf{0}$ for the finite element Stokes equation (2.5). The rest iterates $\mathbf{u}_h^n$ are defined iteratively to be the unique solution of

$$a(\mathbf{u}_h^n, \mathbf{v}_h) + \alpha(\operatorname{div}\mathbf{u}_h^n, \operatorname{div}\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) - (\operatorname{div}\sum_{j=0}^{n-1}\alpha\mathbf{u}_h^j, \operatorname{div}\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{2.9}$$

$n = 1, 2, \cdots$. Here α is positive constant. At the end of iteration, we let

$$p_h^n = -\operatorname{div}\sum_{j=0}^{n}\alpha\mathbf{u}_h^j.$$

∎

Remark 2.1 *In the iterated penalty method of Definition 2.1, we need only to do computer coding for the continuous $Q_{k+1,k} \times Q_{k,k+1}$ element for the vector Laplacian-like equations.*

Remark 2.2 *By Definition 2.1, at the convergence of iteration,* $\operatorname{div} \mathbf{u}_h^n = 0$ *and we obtain the solution* $\mathbf{u}_h$ *of (2.5). Consequently, the unique solution* p_h *of (2.5) is obtained as a byproduct. We note that the spurious (checkerboard) modes, generated at each singular vertex (cf. (3.8)), in the discrete pressure space are filtered out naturally. There are other stabilization techniques, when singular or near-singular vertices are presented, cf. [12].*

Theorem 2.2 *(Theorem 1.1.19 in [5].) Let the inf-sup condition (3.1) hold. The iterated penalty method (2.9) converges with a constant rate independent of the grid size h, i.e.,*

$$\|\mathbf{e}^{n+1}\|_{H^1(\Omega)^2} \le C\alpha^{-1}\|\mathbf{e}^n\|_{H^1(\Omega)^2},$$

where $\mathbf{e}^n = \mathbf{u}_h - \mathbf{u}_h^n$.

Proof. This theorem is proved as Theorem 1.1.19 in [5]. For completeness, we present the proof in our special case, slightly simpler than the original presentation. By (2.5) and (2.9),

$$a(\mathbf{e}^{n+1}, \mathbf{v}_h) + \alpha(\operatorname{div} \mathbf{e}^{n+1}, \operatorname{div} \mathbf{v}_h) = (p_h - p_h^n, \operatorname{div} \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{2.10}$$

By (2.4) and (3.1), as $\operatorname{div} \mathbf{e}^{n+1} \in P_h$, there is a $\mathbf{w}_h \in \mathbf{V}_h$ such that

$$\operatorname{div} \mathbf{w}_h = \operatorname{div} \mathbf{e}^{n+1}, \tag{2.11}$$

$$\|\mathbf{w}_h\|_{H^1} \le C\|\operatorname{div} \mathbf{e}^{n+1}\|_{L^2}. \tag{2.12}$$

By (2.9), we have also

$$\begin{aligned} a(\mathbf{u}_h^n, \mathbf{v}_h) &= (f, \mathbf{v}_h) + (p_h^n, \operatorname{div} \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ a(\mathbf{e}_h^n, \mathbf{v}_h) &= (p_h - p_h^n, \operatorname{div} \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \end{aligned} \tag{2.13}$$

Therefore, by (2.10) and (2.13),

$$|\mathbf{e}^{n+1}|_{H^1}^2 + \alpha\|\operatorname{div} \mathbf{e}^{n+1}\|_{L^2}^2 = a(\mathbf{e}^n, \mathbf{e}^{n+1}) \le |\mathbf{e}^n|_{H^1}|\mathbf{e}^{n+1}|_{H^1}. \tag{2.14}$$

By Lemma 2.1 of [21], i.e., $a(\mathbf{u}_h, \mathbf{v}_h) = (\operatorname{div} \mathbf{u}_h, \operatorname{div} \mathbf{v}_h) + (\operatorname{curl} \mathbf{u}_h, \operatorname{curl} \mathbf{v}_h)$, (2.10) becomes to, by letting $\mathbf{v}_h = \mathbf{w}_h$,

$$-\|\operatorname{curl} \mathbf{e}^{n+1}\|_{L^2}||\operatorname{curl} \mathbf{w}_h||_{L^2} + (1 + C\alpha)||\operatorname{div} \mathbf{e}^{n+1}\|_{L^2}^2 \le |(p_h - p_h^n, \operatorname{div} \mathbf{w}_h)|. \tag{2.15}$$

By (2.15), (2.13) and (2.12), we get

$$\begin{aligned} (1 + C\alpha)\|\operatorname{div} \mathbf{e}^{n+1}\|_{L^2}^2 &\le |\mathbf{e}^n|_{H^1}|\mathbf{w}_h|_{H^1} + \|\operatorname{curl} \mathbf{e}^{n+1}||_{L^2}\|\operatorname{curl} \mathbf{w}_h\|_{L^2}, \\ &\le (C|\mathbf{e}^n|_{H^1}\| + C\|\operatorname{curl} \mathbf{e}^{n+1}\|_{L^2})\|\operatorname{div} \mathbf{e}^{n+1}\|_{L^2} \\ &\le 2C|\mathbf{e}^n|_{H^1}\|\operatorname{div} \mathbf{e}^{n+1}\|_{L^2}, \end{aligned}$$

where in the last step, we applied (2.14) again. Hence,

$$\|\operatorname{div} \mathbf{e}^{n+1}||_{L^2} \le C\alpha^{-1}|\mathbf{e}^n|_{H^1}. \tag{2.16}$$

Let $\mathbf{w}_h$ be defined in (2.11) and (2.12). Let $\mathbf{z}_h \in \mathbf{Z}_h$ be $a(\cdot,\cdot)$ projection of $(\mathbf{e}^{n+1}-\mathbf{w}_h)$, defined by

$$a(\mathbf{z}_h, \mathbf{v}_h) = a(\mathbf{e}^{n+1} - \mathbf{w}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{Z}_h. \tag{2.17}$$

Then $(\mathbf{e}^{n+1} - \mathbf{w}_h - \mathbf{z}_h) \in \mathbf{Z}_h^{\perp_a}$. But, by (2.11), $\operatorname{div}(\mathbf{e}^{n+1} - \mathbf{w}_h) = 0$, i.e., $(\mathbf{e}^{n+1} - \mathbf{w}_h) \in \mathbf{Z}_h$. Since $\mathbf{z}_h \in \mathbf{Z}_h$, we have also $(\mathbf{e}^{n+1} - \mathbf{w}_h - \mathbf{z}_h) \in \mathbf{Z}_h$. We conclude that

$$\mathbf{e}^{n+1} - \mathbf{w}_h - \mathbf{z}_h = \mathbf{0}. \tag{2.18}$$

By (2.10),

$$a(\mathbf{e}^{n+1}, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{Z}_h.$$

Then, by (2.17), we find out that $\mathbf{z}_h$ is also an $a(\cdot,\cdot)$ projection of $-\mathbf{w}_h$ in a subspace $\mathbf{Z}_h$:

$$a(\mathbf{z}_h, \mathbf{v}_h) = a(-\mathbf{w}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{Z}_h.$$

So $|\mathbf{z}_h|_{H^1} \le |\mathbf{w}_h|_{H^1}$. Combining (2.18), (2.12) and (2.16), we prove the theorem as

$$|\mathbf{e}^{n+1}|_{H^1} = |\mathbf{w}_h + \mathbf{z}_h|_{H^1} \le 2|\mathbf{w}_h|_{H^1} \le C\|\operatorname{div} \mathbf{e}^{n+1}\|_{L^2} \le C\alpha^{-1}|\mathbf{e}^n|_{H^1}.$$

∎

We next give a rough count on the number of arithmetic operations for the iterated penalty method. We show that the method is of optimal order, i.e., solving the linear system (2.5) of N unknowns within CN operations for some constant C independent of N, but dependent on the polynomial degree k and the uniform inf-sup constant in (3.1). First, for a given h and k, let γ be the truncation (discretization) error defined in (3.18). By Theorem 2.2, selecting a fixed α according to the inf-sup constant in (3.1), we need to do m steps of iterated penalty method to reach

$$\|\mathbf{u}_h - \mathbf{u}_h^m\|_A \le \gamma/2, \tag{2.19}$$

where $\|\cdot\|_A^2 = |\cdot|_{H^1}^2 + \alpha\|\cdot\|_{L^2}^2$. In (2.19), m is independent of h, i.e., the level of multigrid refinement. We note that γ depends on h, but m does not. m depends on the inf-sup condition only, independent of k and h. To be accurate, we need to use the multigrid solutions on coarse levels as initial guesses to avoid an h-dependence, cf. [3]. We skip such technical but also purely theoretical details.

Let $\hat{\mathbf{u}}_h^n$ be the multigrid iterative solution to (2.9), up to the error $2^{-m-1}\gamma$. To be precise, $\hat{\mathbf{u}}_h^n$ is an approximation, recursively defined by the earlier inaccurate solutions, to the solution of the following equation

$$a(\bar{\mathbf{u}}_h^n, \mathbf{v}_h) + \alpha(\operatorname{div} \bar{\mathbf{u}}_h^n, \operatorname{div} \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) - (\operatorname{div} \sum_{j=0}^{n-1} \alpha\hat{\mathbf{u}}_h^j, \operatorname{div} \mathbf{v}_h),$$

such that

$$\|\bar{\mathbf{u}}_h^n - \hat{\mathbf{u}}_h^n\|_A \le 2^{-m-1}\gamma.$$

We analyze the iterative solutions inductively. As we choose $\mathbf{u}_h^0 = \mathbf{0}$, we have $\bar{\mathbf{u}}_h^1 = \mathbf{u}_h^1$ and

$$\|\mathbf{u}_h^1 - \hat{\mathbf{u}}_h^1\|_A \le 2^{-m-1}\gamma.$$

Next, we solve the following equation to get $\hat{\mathbf{u}}_h^2$

$$a(\bar{\mathbf{u}}_h^2, \mathbf{v}_h) + \alpha(\operatorname{div} \bar{\mathbf{u}}_h^2, \operatorname{div} \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) - (\operatorname{div} \alpha \hat{\mathbf{u}}_h^1, \operatorname{div} \mathbf{v}_h). \tag{2.20}$$

By Theorem 2.2,

$$\begin{aligned}||\mathbf{u}_h^2 - \hat{\mathbf{u}}_h^2\|_A &\le \|\mathbf{u}_h^2 - \bar{\mathbf{u}}_h^2\|_A + \|\bar{\mathbf{u}}_h^2 - \hat{\mathbf{u}}_h^2||_A \le \sqrt{\alpha}\| \operatorname{div}(\mathbf{u}_h^1 - \hat{\mathbf{u}}_h^1)\|_{L^2} + \|\bar{\mathbf{u}}_h^2 - \hat{\mathbf{u}}_h^2\|_A \\ &\le \|\mathbf{u}_h^1 - \hat{\mathbf{u}}_h^1\|_A + 2^{-m-1}\gamma \le (1+1)2^{-m-1}\gamma.\end{aligned}$$

Inductively, we get that

$$\begin{aligned}\|\mathbf{u}_h^m - \hat{\mathbf{u}}_h^m\|_A &\le \|\mathbf{u}_h^m - \bar{\mathbf{u}}_h^m||_A + \|\bar{\mathbf{u}}_h^m - \hat{\mathbf{u}}_h^m\|_A \le \sum_{j=1}^{m-1} \|\mathbf{u}_h^j - \hat{\mathbf{u}}_h^j\|_A + 2^{-m-1}\gamma \\ &\le (1 + (1+1) + (1+2+1) + (1+2+4+1) + \cdots + 1)2^{-m-1}\gamma \\ &= 2^m 2^{-m-1}\gamma = \gamma/2.\end{aligned}$$

Therefore, by mCN operations (each equation (2.20) needs CN operations in the multigrid method, cf. [3]), the iterated penalty method combined with the multigrid solve would produce a solution $\hat{\mathbf{u}}_h^m$ up to the optimal order of truncation error:

$$\begin{aligned}|\mathbf{u}_h - \hat{\mathbf{u}}_h^m|_{H^1} &\le \|\mathbf{u}_h - \hat{\mathbf{u}}_h^m\|_A \\ &\le \|\mathbf{u}_h - \mathbf{u}_h^m\|_A + \|\mathbf{u}_h^m - \hat{\mathbf{u}}_h^m||_A \le \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma.\end{aligned}$$

3 Convergence theory

The central task is to prove the inf-sup condition, or the LBB condition ([14, 6]), for the discrete finite element spaces:

$$\inf_{0\ne q\in P_h} \sup_{\mathbf{v}\in\mathbf{V}_h} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{H^1}\|q\|_{L^2}} \ge C, \tag{3.1}$$

for some constant $C > 0$ independent of the grid size h. We will construct a velocity $\mathbf{v}$ in three steps for a given discrete pressure $q \in P_h$. $\mathbf{v}_1$ will matches the integrals $\int_K q$ on each element, constructed globally. $\mathbf{v}_2$ will be constructed locally so that div $\mathbf{v}_2$ match $q - \operatorname{div} \mathbf{v}_1$ at all vertices, within each element. $\mathbf{v}_3$ will be constructed element-wise to match the difference $q - \operatorname{div}(\mathbf{v}_1 + \mathbf{v}_2)$.

Lemma 3.1 *The finite element triple $\{\hat{K}, \Sigma_{\hat{K}}, Q_{3,2}(\hat{K})\}$ is well defined, where the set of functionals $\Sigma_{\hat{K}}$ is (see Figure 1)*

$$\Sigma_{\hat{K}} = \{f \mid f(v): \ v(\frac{-1+2i}{3}, \pm 1), 0 \le i \le 3; \ v(\pm\frac{1}{3}, 0); \ \int_{-1}^{1} v(\pm 1, y)dy\}. \tag{3.2}$$

Proof. We note that we alter only two boundary degrees of freedom of the standard C^0-$Q_{3,2}$ finite element, cf. Ciarlet [9], from a mid-edge function value to an integral on the edge. We only need to show the uni-solvency of $\Sigma_{\hat{K}}$ as we have a finite dimensional problem. Let $v \in Q_{3,2}$, a polynomial on $\hat{K}$ such that $f(v) = 0$ for the all 12 functionals in (3.2), depicted in Figure 1.

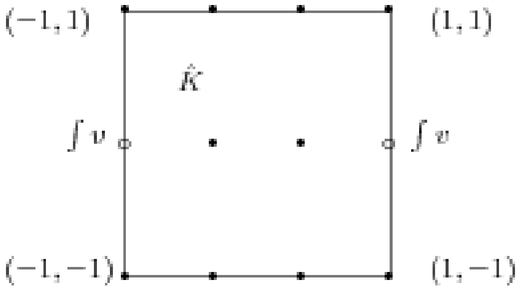

Figure 1: The set of linear functionals $\Sigma_{\hat{K}}$, cf. (3.2).

When restricted on an edge $y = 1$, v is a degree 3 polynomial in x and vanishes at 4 points on the edge: $(-1, 1), (-1/3, 1), (1/3, 1)$ and $(1, 1)$. So $v \equiv 0$ on $y = 1$ and $v(x, y) = (y-1)v_1(x, y)$ for some $Q_{3,1}$ polynomial $v_1(x, y)$. Repeating this argument on the edge $y = -1$ of $\hat{K}$, we have

$$v = (y^2 - 1)v_2(x),$$

where $v_2(x)$ is a cubic polynomial in x. As $v = 0$ at two points on line $y = 0$ (see Figure 1),

$$v = (y^2 - 1)(ax + b)(x + \frac{1}{3})(x - \frac{1}{3})$$

for some constants a and b. Finally, $\int v = 0$ on the two vertical edges. By which, we get

$$\int_{-1}^{1} v_x(\pm 1, y)dy = \frac{8}{9}(b \pm a)\int_{-1}^{1}(y^2 - 1)dy = -\frac{32}{27}(b \pm a) = 0.$$

Thus $a = b = 0$, and $v \equiv 0$. ∎

By Lemma 3.1, we have a dual basis for $Q_{3,2}$ polynomials on $\hat{K}$ corresponding to the linear functional set $\Sigma_{\hat{K}}$, i.e., $\{\hat{\phi}_i\}_{i=1}^{12}$, such that $f_j(\hat{\phi}_i) = \delta_{ji}$. Following the standard routine, we can define a global basis $\{\phi_i\}$ on the grid Ω_h via the referencing mappings defined on each element $K \in \Omega_h$. The linear span of such global basis functions would define the C^0-$Q_{3,2}$ space on Ω_h:

$$V_{3,2} = \left\{ u_h \in C^0(\Omega) \;\middle|\; u_h|_{\Omega} = 0,\ u_h|_K \in Q_{3,2}\ \forall K \in \Omega_h \right\}.$$

For any C^0-$Q_{3,2}$ function $u_h(x, y)$, we have

$$u_h(x, y) = \sum f_i(u_h)\phi_i(x, y),$$

where the functional f_i either takes a nodal value of u_h at a nodal point shown in Figure 1, or the integral value of u_h on a vertical edge in the grid Ω_h. The standard nodal value interpolation operator is defined as

$$I_h \ :\ C^0(\Omega) \to V_{3,2}, \quad I_h u = \sum f_i(u_h)\phi_i(x, y). \tag{3.3}$$

Repeating above work, we can define the C^0-$Q_{2,3}$ space $V_{2,3}$, replacing the two vertical-edge integral functionals in (3.2) by two horizontal-edge integrals. Then for $k = 2$, the velocity space in (2.3) is

$$\mathbf{V}_h = V_{3,2} \times V_{2,3}.$$

Lemma 3.2 *For any $q \in P_h$ defined in (2.4), $k \ge 2$, there is a function $\mathbf{v}_1 \in V_{3,2} \times V_{2,3}$ such that*

$$\int_K \operatorname{div} \mathbf{v}_1 = \int_K q \quad \forall K \in \Omega_h, \quad \textit{and} \quad ||\mathbf{v}_1||_{H^1} \le C\|q\|_{L^2}. \tag{3.4}$$

Proof. For any $q \in P_h$, by the theory on regular inversion of the divergence operator developed by Arnold, Scott and Vogelius in [2], there is a $\mathbf{u}_q \in H^{1+\alpha}(\Omega)^2 \cap H_0^1(\Omega)^2$ such that

$$\operatorname{div} \mathbf{u}_q(x,y) = q(x,y) \quad \text{ a. e. for } (x,y) \in \Omega,$$

and

$$||\mathbf{u}_q||_{H^1} \le C||q||_{L^2},$$

for any $\alpha < 1/2$. Let the nodal interpolant be $\mathbf{v}_1 = \mathbf{I}_h \mathbf{u}_q$ where the first component of the interpolation operator $\mathbf{I}_h$ is defined in (3.3). Such a nodal interpolation is stable, cf. [17] for example, that

$$||\mathbf{I}_h \mathbf{u}_q||_{H^1} \le C\|\mathbf{u}_q||_{H^1}.$$

Further, the interpolant also preserve the divergence element-wise:

$$\begin{aligned}
\int_K \operatorname{div} \mathbf{v}_1 dxdy &= \int_{y_{j-1}}^{y_j} [v_1^{(1)}(x_i,y) - v_1^{(1)}(x_{i-1},y)]dy + \int_{x_{i-1}}^{x_i} [v_1^{(2)}(x,y_j) - v_1^{(2)}(x,y_{j-1})]dx \\
&= \int_{y_{j-1}}^{y_j} [u_q^{(1)}(x_i,y) - u_q^{(1)}(x_{i-1},y)]dy + \int_{x_{i-1}}^{x_i} [u_q^{(2)}(x,y_j) - u_q^{(2)}(x,y_{j-1})]dx \\
&= \int_K \operatorname{div} \mathbf{u}_q dxdy = \int_K q dxdy,
\end{aligned}$$

where $K = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ is an element in Ω_h. Here we use a super index for a component index: $\mathbf{v} = (v^{(1)}, v^{(2)})$. ∎

After matching the integral values of q element wise by $\operatorname{div} \mathbf{v}_1$, we next match the vertex-values of $q - \operatorname{div} \mathbf{v}_1$.

Lemma 3.3 *For any $q \in P_h$ defined in (2.4) such that $\int_K q = 0 \quad \forall K \in \Omega_h$, $k \ge 2$, there is a function $\mathbf{v}_2 \in V_{3,2} \times V_{2,3}$ such that*

$$\operatorname{div} \mathbf{v}_2(a_i^K) = q(a_i^K) \quad \forall K \in \Omega_h, \tag{3.5}$$

$$\int_K \operatorname{div} \mathbf{v}_2 = 0 \quad \forall K \in \Omega_h, \tag{3.6}$$

$$||\mathbf{v}_2||_{H^1} \le C\|q\|_{L^2}. \tag{3.7}$$

Here a_i^K, $1 \le i \le 4$, are the four vertices of element K.

Proof. According to Scott and Vogelius [15], all vertices in Ω_h are singular, which means a function $q \in P_h$ is subject to minor continuity constraints:

$$\sum_{i=1}^{4} (-1)^i q(a_i^{K_i}) = 0, \tag{3.8}$$

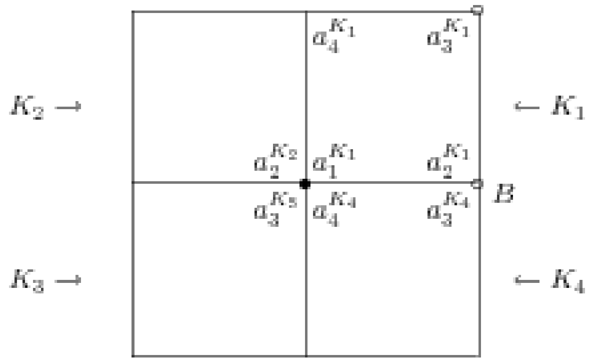

Figure 2: Four vertices $a_i^{K_i}$ meeting at a singular vertex, cf. (3.9).

where $a_i^{K_i}$ are the vertices of four elements K_i meeting at an internal vertex, shown in Figure 2. In particular, at a boundary vertex, see Figure 2, we have

$$q(a_3^{K_1}) = 0, \quad q(a_2^{K_1}) = q(a_3^{K_4}). \tag{3.9}$$

We will construct part of $\mathbf{v}_2$ for the two vertex-values of q at a boundary point B in Figure 2.

Let the element K_1 shown in Figure 2 be $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$. Similarly, we let $K_4 = [x_i, x_{i+1}] \times [y_{j-1}, y_j]$. Let

$$\mathbf{v}_{b0} = \begin{cases} \begin{pmatrix} q(a_2^{K_1}) \frac{(x-x_i)^2(x_{i+1}-x)}{(x_{i+1}-x_i)^2} \frac{(y_{j+1}-y)^2}{(y_{j+1}-y_j)^2} & 0 \end{pmatrix}^T, & (x,y) \in K_1 \\ \begin{pmatrix} q(a_3^{K_4}) \frac{(x-x_i)^2(x_{i+1}-x)}{(x_{i+1}-x_i)^2} \frac{(y-y_{j-1})^2}{(y_j-y_{j-1})^2} & 0 \end{pmatrix}^T, & (x,y) \in K_4 \\ \begin{pmatrix} 0 & 0 \end{pmatrix}^T, & \text{all other } K \in \Omega_h. \end{cases} \tag{3.10}$$

By the condition (3.9), $\mathbf{v}_{b0} \in V_{3,2} \times V_{2,3}$. It can be verified that $\int_K \operatorname{div} \mathbf{v}_{b0} = 0$ on all $K \in \Omega_h$ and that

$$\operatorname{div} \mathbf{v}_{b0}(a) = \begin{cases} q(a) & a = a_2^{K_1}, a_3^{K_4}, \\ 0 & \text{all other vertices } a. \end{cases}$$

We compute the semi-H^1 norm of $\mathbf{v}_{b0}$ in two steps.

$$\int_{K_1} \left(\frac{\partial \mathbf{v}_{b0}}{\partial x} \right)^2 dxdy = \frac{5}{18} q^2(a_2^{K_1})(x_{i+1}-x_i)(y_{j+1}-y_j),$$
$$\int_{K_1} \left(\frac{\partial \mathbf{v}_{b0}}{\partial y} \right)^2 dxdy = \frac{5}{18} q^2(a_2^{K_1})(x_{i+1}-x_i)^2.$$

On the other side, the L^2 norm and L^∞ norm are equivalent on the reference square $\hat{K}$ for the space of polynomials Q_k. There is a constant C (depending on polynomial degree k only) such that

$$\int_{\hat{K}} q_k^2 dxdy \geq C q_k^2(-1,1), \quad \forall q_k \in Q_k.$$

After being scaled by the reference mapping,

$$
\begin{aligned}
|\mathbf{v}_{b0}|^2_{H^1(K_1)^2} &= \left(1+\frac{x_{i+1}-x_i}{y_{j+1}-y_j}\right) q^2(a_2^{K_1})(x_{i+1}-x_i)(y_{j+1}-y_j) \\
&\le \left(1+\frac{x_{i+1}-x_i}{y_{j+1}-y_j}\right) \frac{1}{C}\|q\|^2_{L^2(K_1)}.
\end{aligned}
$$

Summing over the two elements K_1 and K_4 in Figure 2, it follows that

$$
\|\mathbf{v}_{b0}\|_{H^1(\Omega)^2} \le C\|q\|_{L^2(\Omega)}, \tag{3.11}
$$

where the constant C depends on k and the maximum aspect ratio of grids (independent of h.) Next, at an internal vertex, $a_1^{K_1}$ in Figure 2, we need do the above construction three times:

$$
\begin{aligned}
\mathbf{v}_{b1} &= q(a_3^{K_3})\mathbf{w}_{2,3} && \text{(supported on } K_3 \text{ and } K_2\text{)}, \\
\mathbf{v}_{b2} &= \left(q(a_2^{K_2}) - q(a_3^{K_3})\right)\mathbf{w}_{1,2} && \text{(supported on } K_2 \text{ and } K_1\text{)}, \\
\mathbf{v}_{b3} &= \left(q(a_1^{K_1}) - q(a_2^{K_2}) + q(a_3^{K_3})\right)\mathbf{w}_{1,4} && \text{(supported on } K_1 \text{ and } K_4\text{)} \\
&= q(a_4^{K_4})\mathbf{w}_{1,4} && \text{(by (3.8))},
\end{aligned}
$$

where $\mathbf{w}_{i,j}$ are special functions similar to that in (3.10). Here the second components of $\mathbf{w}_{2,3}$ and $\mathbf{w}_{1,4}$ are zero while the first one of $\mathbf{w}_{1,2}$ is zero. We summer all such $\mathbf{v}_{bi}$ to define $\mathbf{v}_2$ satisfying the requirements of the lemma. Since we construct at most 2 nonzero $\mathbf{v}_{bi}$ at each vertex,

$$
\|\mathbf{v}_2\|_{H^1(\Omega)^2} \le 2\sqrt{2}C\|q\|_{L^2(\Omega)},
$$

where C is defined in (3.11). ∎

After we match the element-integrals and the vertex-values of $q \in P_h$, we will next match q point wise on each element.

Lemma 3.4 *Let $q \in P_h$ defined in (2.4) with $k \ge 2$ such that $\int_K q = 0 \quad \forall K \in \Omega_h$ and q vanishes at all vertices of grid Ω_h. There is a function $\mathbf{v}_3 \in \mathbf{V}_h$ such that*

$$
\operatorname{div} \mathbf{v}_3(x,y) = q(x,y) \quad \forall (x,y) \in \Omega, \tag{3.12}
$$

$$
\mathbf{v}_3(x,y) = 0 \quad \forall (x,y) \in \partial K \text{ and } \forall K \in \Omega_h, \tag{3.13}
$$

$$
\|\mathbf{v}_3\|_{H^1} \le C\|q\|_{L^2}. \tag{3.14}
$$

Proof. We first prove the lemma for $k=2$. The construction is needed only on the reference element $\hat K$. We then map constructed $\mathbf{v}_3$ to each element K as the support of $\mathbf{v}_3$ is isolated, $\mathbf{v}_3|_{\partial K} = 0$. Let $\hat q \in Q_2(\hat K)$ such that $\hat q(\pm1,\pm1) = 0$. Let

$$
\begin{aligned}
\hat{\mathbf{v}}_3 = \frac{1}{4}(1-x^2)(1-y^2)\Big[&\hat q(-1,0)\begin{pmatrix}1-x\\0\end{pmatrix} - \hat q(1,0)\begin{pmatrix}1+x\\0\end{pmatrix} \\
&+ \hat q(0,-1)\begin{pmatrix}0\\1-y\end{pmatrix} - \hat q(0,1)\begin{pmatrix}0\\1+y\end{pmatrix}\Big].
\end{aligned}
$$

$\hat{\mathbf{v}}_3 \in C^0\text{-}Q_{3,2} \times Q_{2,3}$. By the construction, $\hat q - \operatorname{div} \hat{\mathbf{v}}_3 = 0$ on the boundary $\hat K$, i.e.,

$$
\hat q - \operatorname{div} \hat{\mathbf{v}}_3 = c(1-x^2)(1-y^2).
$$

By the given condition $\int_K q = 0$, we conclude that $c = 0$, and

$$\operatorname{div} \hat{\mathbf{v}}_3 = \hat{q}.$$

By such an element-wise construction, we obtain an $\mathbf{v}_3$ such that $\operatorname{div} \mathbf{v}_3 = q$. By the same scaling argument used in Lemma 3.3, it follows that

$$\|\mathbf{v}_3\|_{H^1} \leq C\|q\|_{L^2}. \tag{3.15}$$

The lemma is proven for $k = 2$. We note that the case $k = 2$ is special as the subset

$$P_{h0} := \left\{ q \in P_h \mid \int_K q = 0 \ \text{ and } q|_{\partial K} = 0 \quad \forall K \in \Omega_h \right\} \tag{3.16}$$

of functions $q \in P_h$ have no internal degree of freedom on each element.

For $k > 2$, the construction for $\mathbf{v}_3$ on the boundary of each element K remains the same. Let $\mathbf{v}_{30} \in \mathbf{V}_h$ such that, by (3.15) and (3.16),

$$q - \operatorname{div} \mathbf{v}_{30} \in P_{h0} \ \text{ and } \ \|\mathbf{v}_{30}\|_{H^1} \leq C\|q\|_{L^2}.$$

But for $k > 2$, when matching the internal values of $(q - \operatorname{div} \mathbf{v}_{30})$, we have extra freedom in defining $\mathbf{v}_{31}$ inside each element K. The reason is that we have internal divergence-free modes, see Figure 3. For low k, such as $k = 3$ and $k = 4$, it is not difficult to use a symbolic software to generate a basis $\{p_{h,i}\}$ for P_{h0} in (3.16) and a corresponding basis $\{\mathbf{v}_{h,i}\}$, where $\operatorname{div} \mathbf{v}_{h,i} = p_{h,i}$ and $\operatorname{curl} \mathbf{v}_{h,i} = 0$. Instead of constructing such $q_{h,i}$ and $\mathbf{v}_{h,i}$ for all degree of polynomial k, we provide a combinatorial argument to prove the lemma.

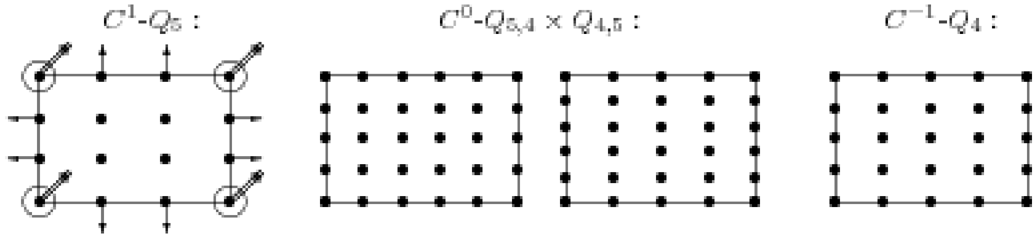

Figure 3: Internal degrees of freedom when $k = 4$.

Let us consider the construction on the reference element $\hat{K}$. The dimension of Q_k polynomials is $(k+1)^2$. When restricted to zero on the boundary of $\hat{K}$, the dimension is $(k-1)^2$. With a further restriction $\int_{\hat{K}} q = 0$, denoting the space by $P_{h0}(\hat{K})$, the dimension is

$$\dim P_{h0}(\hat{K}) = (k-1)^2 - 1.$$

Now, the dimension of $Q_{k+1,k} \times Q_{k,k+1}$ polynomials is $2(k+2)(k+1)$. With zero boundary condition on $\hat{K}$, denoting the space by $\mathbf{V}_h(\hat{K})$, the dimension is reduced to

$$\dim \mathbf{V}_h(\hat{K}) = 2k(k-1).$$

Further, restricting $\partial v^{(1)}/\partial x = 0$ on the two vertical edges and $\partial v^{(2)}/\partial y = 0$ on the two horizontal edges, we have

$$\dim \mathbf{V}_{h0}(\hat{K}) = 2(k-2)(k-1),$$

where

$$\mathbf{V}_{h0}(\hat K) = \left\{ \mathbf{v} \in \mathbf{V}_h(\hat K) \mid \operatorname{div} \mathbf{v} \in P_{h0}(\hat K) \right\}.$$

The kernel of div on $\mathbf{V}_{h0}(\hat K)$ is the curl of Q_{k+1} polynomials with homogeneous boundary conditions:

$$S := \left\{ s \in Q_{k+1}(\hat K) \mid s = 0 \text{ and } \partial s/\partial \mathbf{n} = 0 \text{ on } \partial \hat K \right\}.$$

Since the kernel of curl on S is $\{0\}$, we have

$$\begin{aligned}\dim(\operatorname{div} \mathbf{V}_{h0}(\hat K)) &= \dim \mathbf{V}_{h0}(\hat K) - \dim S = 2(k-2)(k-1) - (k-2)^2 \\ &= (k-2)(2k-2-k+2) = (k-2)k = (k-1)^2 - 1 = \dim P_{h0}.\end{aligned}$$

Thus div is an on-to mapping from $\mathbf{V}_{h0}(\hat K)$ to $P_{h0}(\hat K)$. For each $q_0 \in P_{h0}(\hat K)$, there is a $\mathbf{v}_0 \in \mathbf{V}_{h0}(\hat K)$, after an L^2 orthogonal project to the complement of S in $\mathbf{V}_{h0}(\hat K)$, such that

$$\operatorname{div} \mathbf{v}_0 = q_0 \quad \text{and } \operatorname{curl} \mathbf{v}_0 = 0.$$

Therefore,

$$|\mathbf{v}_0|^2_{H^1(\Omega)^2} = \|\operatorname{div} \mathbf{v}_0\|^2_{L^2(\Omega)} + \|\operatorname{curl} \mathbf{v}_0\|^2_{L^2(\Omega)} = \|q_0\|^2_{L^2(\Omega)}.$$

By the Poincaré inequality, $\|\mathbf{v}_0\|_{H^1(\Omega)^2} \le C\|q_0\|_{L^2(\Omega)}$. Finally, repeating such a construction on each element, we find a $\mathbf{v}_{h0}$, supported inside all elements of Ω_h, such that

$$\operatorname{div} \mathbf{v}_{h0} = q - \operatorname{div} \mathbf{v}_{30} \quad \text{and that} \quad \|\mathbf{v}_{h0}\|_{H^1} \le C\|q - \operatorname{div} \mathbf{v}_{30}\|_{L^2}.$$

Let $\mathbf{v}_3 = \mathbf{v}_{30} + \mathbf{v}_{h0}$. The lemma is proven. ∎

Theorem 3.1 *Let $k \ge 2$. The mixed finite element pair $(\mathbf{V}_h, P_h)$ defined in (2.3) and (2.4) is stable, i.e., (3.1) holds.*

Proof. For any $q \in P_h$, we construct a $\mathbf{v} \in \mathbf{V}_h$ to satisfy (3.1). By (3.4), there is a $\mathbf{v}_1 \in \mathbf{V}_h$ such that

$$\begin{aligned}\int_K (q - \operatorname{div} \mathbf{v}_1) = 0 \quad \forall K \in \Omega_h, \\ \|\mathbf{v}_1\|_{H^1(\Omega)^2} \le C_1\|q\|_{L^2(\Omega)}.\end{aligned}$$

By (3.5), there is a $\mathbf{v}_2 \in \mathbf{V}_h$ such that

$$\begin{aligned}[\operatorname{div} \mathbf{v}_2 - (q - \operatorname{div} \mathbf{v}_1)]\big|_K (a_i) = 0 \quad \forall K \in \Omega_h \text{ and for all vertices of } K, \\ \|\mathbf{v}_2\|_{H^1(\Omega)^2} \le C_2\|q - \operatorname{div} \mathbf{v}_1\|_{L^2(\Omega)}.\end{aligned}$$

By (3.12), there is a $\mathbf{v}_3 \in \mathbf{V}_h$ such that

$$\begin{aligned}\operatorname{div} \mathbf{v}_3 = q - \operatorname{div} \mathbf{v}_1 - \operatorname{div} \mathbf{v}_2, \\ \|\mathbf{v}_3\|_{H^1(\Omega)^2} \le C_3\|q - \operatorname{div} \mathbf{v}_1 - \operatorname{div} \mathbf{v}_2\|_{L^2(\Omega)}.\end{aligned}$$

Let $\mathbf{v} = -\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3$. It follows that

$$\begin{aligned}
\|\mathbf{v}\|_{H^1(\Omega)^2} &\le \|\mathbf{v}_1\|_{H^1(\Omega)^2} + \|\mathbf{v}_2\|_{H^1(\Omega)^2} + \|\mathbf{v}_3\|_{H^1(\Omega)^2} \\
&\le C_1\|q\|_{L^2(\Omega)} + C_2\|q - \operatorname{div}\mathbf{v}_1\|_{L^2(\Omega)} + C_3\|q - \operatorname{div}\mathbf{v}_1 - \operatorname{div}\mathbf{v}_2\|_{L^2(\Omega)} \\
&\le C_1\|q\|_{L^2(\Omega)} + C_2(\|q\|_{L^2(\Omega)} + \|\mathbf{v}_1\|_{H^1(\Omega)^2}) \\
&\quad + C_3(\|q\|_{L^2(\Omega)} + \|\mathbf{v}_1\|_{H^1(\Omega)^2} + \|\mathbf{v}_2\|_{H^1(\Omega)^2}) \\
&\le (C_1 + C_2 + C_3 + C_1C_2 + C_1C_3 + C_2C_3)\|q\|_{L^2(\Omega)}
\end{aligned}$$

and that

$$\begin{aligned}
b(\mathbf{v}, q) &= (-\operatorname{div}\mathbf{v}, q) = \|q\|^2_{L^2(\Omega)} \\
&\ge (C_1 + C_2 + C_3 + C_1C_2 + C_1C_3 + C_2C_3)^{-1}\|\mathbf{v}\|_{H^1(\Omega)^2}\|q\|_{L^2(\Omega)}.
\end{aligned}$$

(3.1) is proved with $C = (C_1 + C_2 + C_3 + C_1C_2 + C_1C_3 + C_2C_3)^{-1}$. ∎

Corollary 3.1 *Let $k \ge 2$. The discrete pressure space (2.4) is characterized by*

$$P_h = \left\{ q \mid q|_K \in Q_k \ \forall K \in \Omega_h; \int_\Omega q = 0; \sum_{i=1}^{i_0} (-1)^i q(a_i^{K_i}) = 0 \right\}, \tag{3.17}$$

where $a_i^{K_i}$ are defined in (3.8), and $i_0 = 4$ at an internal vertex, but i_0 may equal to 1 or 2 at the boundary.

Proof. In the proof of Theorem 3.1, we constructed a $\mathbf{v} = -\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3$ for any q_h in the set of (3.17), such that $\operatorname{div}\mathbf{v} = -q_h$. Thus the P_h defined in (3.17) is a subset of that P_h defined in (2.4). On the other side, by the proof of Lemma 3.3, we known that $\operatorname{div}\mathbf{v}_h$ satisfies the constraints in (3.17) for all $\mathbf{v}_h \in \mathbf{V}_h$. ∎

Theorem 3.2 *Let $k \ge 2$. The discrete solution $(\mathbf{u}_h, p_h)$ of (2.5) approximate that of (2.2) in the optimal order:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^1(\Omega)^2} + \|p - p_h\|_{L^2(\Omega)} \le Ch^{\min\{k,r\}}(\|\mathbf{u}\|_{H^{r+1}(\Omega)^2} + \|p\|_{H^r(\Omega)}), \quad r \ge 1. \tag{3.18}$$

Proof. By the inf-sup condition (3.1) and the standard mixed finite element theory [14], it follows that

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_{H^1(\Omega)^2} + \|p - p_h\|_{L^2(\Omega)} &\le C(\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{H^1(\Omega)^2} + \inf_{q_h \in P_h} \|p - q_h\|_{L^2(\Omega)}) \\
&\le (\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{H^1(\Omega)^2} + \inf_{q_h \in \tilde{P}_h} \|p - q_h\|_{L^2(\Omega)})
\end{aligned}$$

where $\tilde{P}_h$ is the space of continuous Q_k polynomials with mean value zero:

$$\tilde{P}_h = \left\{ q_h \in C^0(\Omega) \mid \int_\Omega q_h = 0, \ q_h|_K \in Q_k \ \forall K \in \Omega_h \right\}.$$

We note that $\tilde{P}_h \subset P_h$, by (3.17), as functions in $\tilde{P}_h$ satisfy the constraints in (3.17). The theorem is proven as both spaces $\mathbf{V}_h$ and $\tilde{P}_h$ provide the optimal order of approximation. ∎

4 Numerical tests

In this section, we report some results of numerical experiments on the $Q_{k+1,k} \times Q_{k,k+1}$-$Q_k^-$element for the stationary Stokes equations (2.1) on the unit square. We choose a simple exact solution so that the right hand side function $\mathbf{f}$ for (2.1) is

$$\mathbf{f} = -\Delta\, \mathbf{curl}\, g - \nabla g_{xx} = \begin{pmatrix} -g_{yxx} - g_{yyy} - g_{xxx} \\ g_{xxx} + g_{xyy} - g_{yxx} \end{pmatrix}, \tag{4.1}$$

where

$$g = 2^8(x - x^2)^2(y - y^2)^2.$$

The continuous solution for the Stokes equations (2.1) is (see Figure 4)

$$\mathbf{u} = \mathbf{curl}\, g, \quad p = -g_{xx}. \tag{4.2}$$

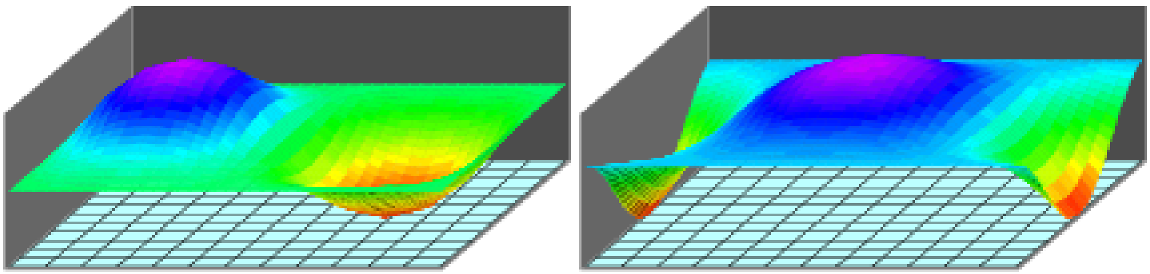

Figure 4: The exact solution, second component of $\mathbf{u}$ and p on the level 2 grid.

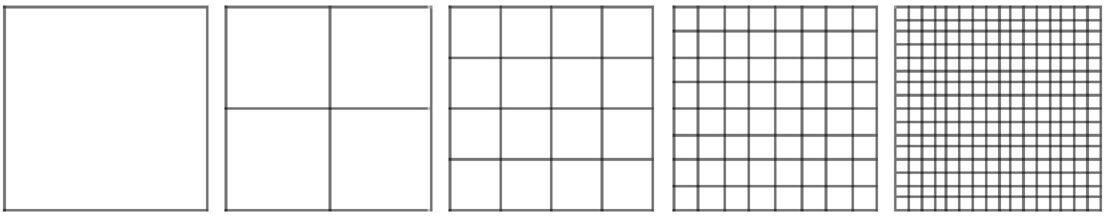

Figure 5: The first five levels of grids.

The grids Ω_h are depicted in Figure 5, i.e., each squares are refined into 4 sub-squares each level. In Table 1 we list various norms and orders of convergence for the finite element solutions in the spaces $\mathbf{V}_h \times P_h$ defined in (2.3)-(2.4). Here we do enough iterated penalty iterations defined in Definition 2.1 until the iterative error is smaller than the truncation error each time.

The H^1 and L^2 errors and convergence orders, reported in Table 1, are consistent with the error bound proved in Theorem 3.2. In Table 1, the nodal errors are of the optimal order too, but not proved in this paper. To be precise, the order of convergence for the velocity when using the $Q_{3,2} \times Q_{2,3}$-Q_2 element, or using the $Q_{4,3} \times Q_{3,4}$-Q_3 element, is one order, or two orders higher than that predicted by the theory, respectively. This might be due to the superconvergence of finite elements, cf. [20]. Or it might be caused by, in addition, the special

Table 1: The errors by spaces $\mathbf{V}_h \times P_h$ defined in (2.3)-(2.4), on Figure 5 grids.

level	$\lvert\mathbf{u}-\mathbf{u}_h\rvert_{H^1}$	h^n	$\lvert\mathbf{u}-\mathbf{u}_h\rvert_{l_\infty}$	h^n	$\lVert p-p_h\rVert_{L^2}$	h^m	$\lVert p-p_h\rVert_{l_\infty}$	h^m
	For the $Q_{3,2} \times Q_{2,3}$-Q_2 element							
2	1.00989		0.1231114		11.024		2.82776	
3	0.10747	3.2	0.0091183	3.8	2.755	2.0	0.46705	2.6
4	0.01274	3.1	0.0005780	4.0	0.671	2.0	0.08399	2.5
5	0.00157	3.0	0.0000369	4.0	0.170	2.0	0.01257	2.7
6	0.00019	3.0	0.0000023	4.0	0.043	2.0	0.00171	2.9
7	0.00002	3.0	0.0000001	4.0	0.011	2.0	0.00022	2.9
	For the $Q_{4,3} \times Q_{3,4}$-Q_3 element							
2	0.014787998		0.0040368767		0.775887	2.0	0.1949967	2.8
3	0.000755147	4.3	0.0000968078	5.4	0.113266	2.8	0.0164941	3.6
4	0.000028532	4.7	0.0000018909	5.7	0.015266	2.9	0.0012285	3.7
5	0.000000966	4.9	0.0000000322	5.9	0.001954	3.0	0.0000838	3.9
6	0.000000031	4.9	0.0000000005	5.9	0.000245	3.0	0.0000055	3.9
7	0.000000001	5.0	0.0000000000	6.0	0.000030	3.0	0.0000003	4.0

Table 2: The errors by the $Q_{5,4} \times Q_{4,5}$-Q_4 element, on Figure 5 grids.

level	$\lvert\mathbf{u}-\mathbf{u}_h\rvert_{H^1}$	h^n	$\lvert\mathbf{u}-\mathbf{u}_h\rvert_{l_\infty}$	h^n	$\lVert p-p_h\rVert_{L^2}$	h^m	$\lVert p-p_h\rVert_{l_\infty}$	h^m
2	0.0000106532	5.3	0.0000010262	5.9	0.0007548	2.8	0.0002390	3.1
3	0.0000000029	11.8	0.0000000002	12.2	0.0000005	10.3	0.0000001	10.8
4	0.0000000000	13.8	0.0000000000	13.8	0.0000000	9.5	0.0000000	10.1

solution (4.2) used in the computation, noting that the convergence orders for the pressure are not higher. We plot the error for the second component of $\mathbf{u}$ on the level 3 grid in Figure 6, for the $Q_{3,2} \times Q_{2,3}$-Q_2 element method. The error of p of the method is shown in Figure 7.

When we increase the polynomial degree k by one to 4, this time, both the exact velocity solution and the exact pressure solution are inside the finite element subspaces, $Q_{5,4} \times Q_{4,5}$-Q_4. Then, on any grid, the numerical solution should be exact, the same as the solution in (4.2), if enough iterated penalty iterations are done and if there is no round-off error. This can be seen in Table 2.

In Table 3, we list the errors and convergence orders from the numerical results when using the $Q_{2,1} \times Q_{1,2}$-Q_1 element. A little surprising, the results are of optimal orders too, but they are not covered by our theory. Further studies are needed to understand and explain the results in Table 3.

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Table 3: The errors by the $Q_{2,1} \times Q_{1,2}$-Q_1 element, on Figure 5 grids.

level	$\lvert\mathbf{u}-\mathbf{u}_h\rvert_{H^1}$	h^n	$\lvert\mathbf{u}-\mathbf{u}_h\rvert_{l_\infty}$	h^n	$\|p-p_h\|_{L^2}$	h^m	$\|p-p_h\|_{l_\infty}$	h^m
3	1.2110203		0.1994708		25.9107521		6.6670602	
4	0.3119692	2.0	0.0457680	2.1	10.5018161	1.3	1.4463214	2.2
5	0.0783616	2.0	0.0110037	2.1	4.9049518	1.1	0.3727838	2.0
6	0.0196081	2.0	0.0027090	2.0	2.4052678	1.0	0.0927318	2.0
7	0.0049030	2.0	0.0006762	2.0	1.1964874	1.0	0.0234598	2.0

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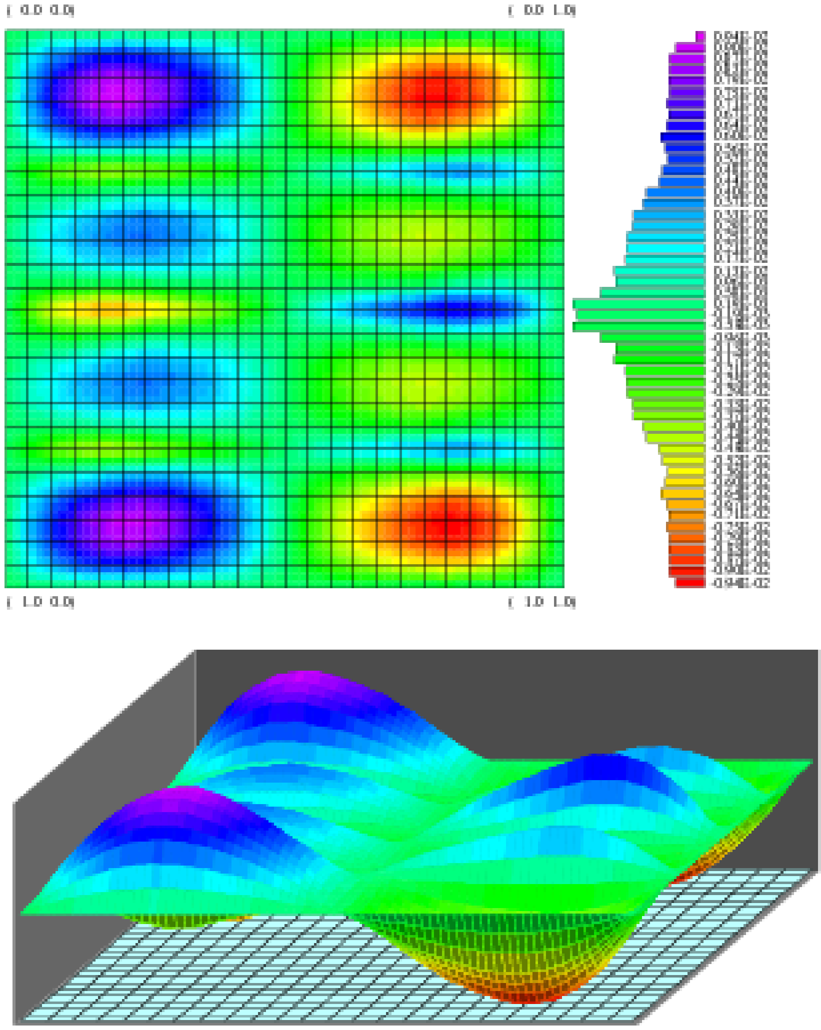

Figure 6: The error of the second component of $\mathbf{u}$ on the level 3 grid ($Q_{3,2} \times Q_{2,3}$-Q_2).

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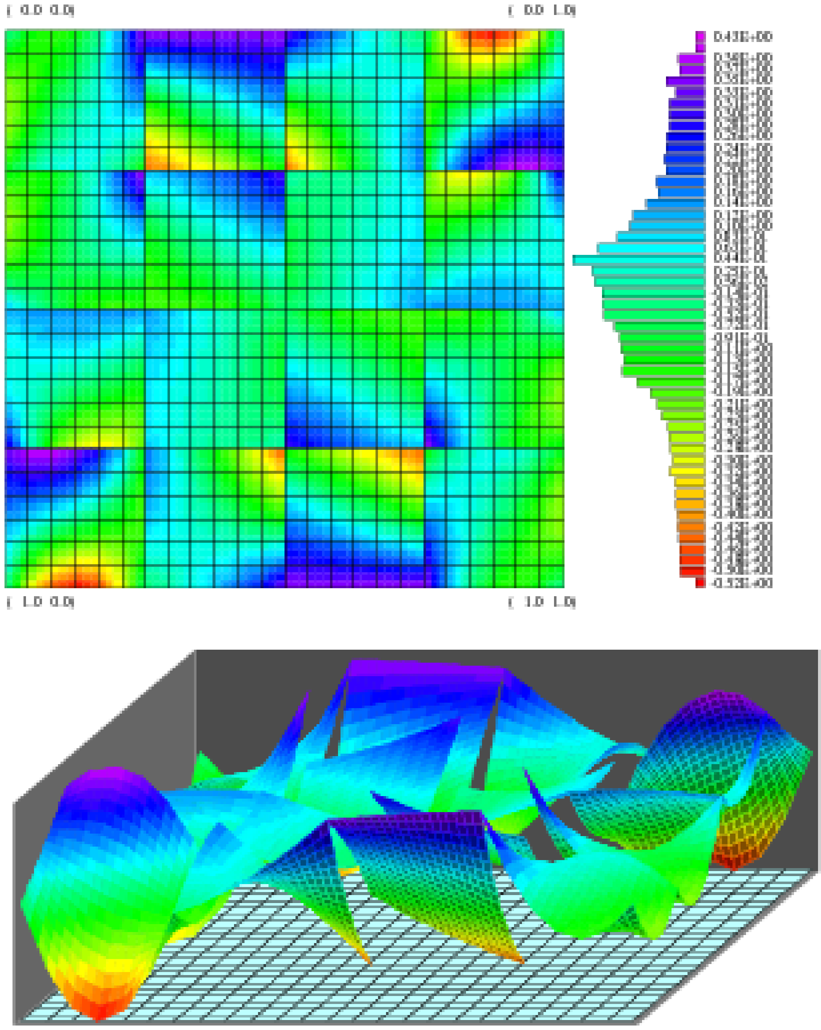

Figure 7: The error of p on the level 3 grid ($Q_{3,2} \times Q_{2,3}$-Q_2).

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