Stability and approximability of the P1-P0 element for Stokes equations

Jinshui Qin* and Shangyou Zhang†

Abstract

In this paper we study the stability and approximability of the \( \mathcal{P}_1-\mathcal{P}_0 \) element (continuous piecewise linear for the velocity and piecewise constant for the pressure on triangles) for Stokes equations. Although this element is unstable for all meshes, it does provide optimal approximations for the velocity and the pressure in many cases. We establish a relation between the stabilities of the \( \mathcal{Q}_1-\mathcal{P}_0 \) element (bilinear/constant on quadrilaterals) and the \( \mathcal{P}_1-\mathcal{P}_0 \) element. We apply many stability results on the \( \mathcal{Q}_1-\mathcal{P}_0 \) element to the analysis of the \( \mathcal{P}_1-\mathcal{P}_0 \) element. We prove that the element has the optimal order of approximations for the velocity and the pressure on a variety of mesh families.

As a byproduct, we also obtain a basis of divergence-free piecewise linear functions on a mesh family on squares. Numerical tests are provided to support the theory and to show the efficiency of the newly discovered, truly divergence-free, \( \mathcal{P}_1 \) finite element spaces in computation.

Keywords. mixed finite elements, Stokes, divergence-free.

1 Introduction

The \( \mathcal{P}_1-\mathcal{P}_0 \) element, continuous piecewise linear approximation for the velocity and piecewise constant approximation for the pressure, is probably the simplest finite element which could preserve the incompressibility condition of incompressible fluids (see [1, 3, 6, 8, 20, 10, 19, 22 and 25] for more information on divergence-free elements for Stokes). Unfortunately, the element is unstable for any mesh since the dimension of the discrete velocity space is always less than that of the pressure space (with Dirichlet boundary condition). However, this element does provide optimal approximations for both the velocity and the pressure on many mesh families. Some discussions on this element can be found in [8, 9, 18 and 23] and references therein. In this paper, we concentrate on the triangular meshes made by crisscross-refinements of quadrilateral meshes, by dividing each quadrilateral of quadrilateral meshes into four triangles by its two diagonals.

To make the notation in introduction clear, we define some notations first. Let \( \mathbf{V}_h \subset \mathbf{H}^1(\Omega) \) and \( P_h \subset L^2(\Omega) := L^2(\Omega)/C \) denote the mixed finite element spaces of velocity and pressure respectively; they are defined on a triangulation \( \mathcal{T}_h \) of a polygonal domain \( \Omega \). We solve the finite element approximation \((\mathbf{u}_h, p_h)\) satisfying the mixed formulation of Stokes equations:

\[
a(\mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p_h) = (f, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h, \\
b(\mathbf{u}_h, q) = 0, \quad \forall q \in P_h.
\]

\[\text{10321 Yellow Pine Dr., Vienna, VA 22182. mathcpsc@aol.com.} \]
\[\text{1Department of Mathematical Sciences, University of Delaware, DE 19716. szhang@udel.edu.} \]
Here \( a(u, v) := \int_\Omega \nabla u : \nabla v, \ b(v, q) := \int_\Omega q \text{div} v, \ \text{and} \ (f, v) := \int_\Omega f \cdot v \) for all \( u, v \in V_h \) and \( q \in P_h \). For convenience, we define \( N_h = \{ q \in P_h | \int_\Omega q \text{div} v = 0, \forall v \in V_h \} \) and the reduced pressure space \( M_h \) as the \( L^2 \)-orthogonal complement of \( N_h \) in \( P_h \). The non constant functions of \( N_h \) are called spurious pressure modes. The element \( V_h \times P_h \) is said to be stable if the inf-sup constant

\[
\gamma_h(V_h, P_h) := \inf_{0 \neq p \in P_h} \sup_{0 \neq v \in V_h} \frac{\int_\Omega p \text{div} v}{\|v\|_{L^2(\Omega)} \|p\|_{L^2(\Omega)}}
\]

is bounded below by a positive number independent of \( h \), see [2] and [7]. Here, \( P_h \) is assumed a subset of \( L_0^p \), i.e., \( \int_\Omega q = 0 \) for all \( q \in P_h \). If \( V_h \times P_h \) is not stable but \( V_h \times M_h \) is, then the element \( V_h \times P_h \) is said to be reduced-stable. Accordingly, the inf-sup constant \( \gamma_h(V_h, M_h) \) is called the reduced inf-sup constant of the element \( V_h \times P_h \).

In this paper, we show a close relation between the stabilities of the \( Q_1-P_0 \) element (bilinear/constant quadrilateral element) and the \( P_1-P_0 \) element. That is, the reduced stability of the \( Q_1-P_0 \) element on a family of quadrilateral meshes is equivalent to that for \( P_1-P_0 \) on the crisscross-refinement meshes of the quadrilateral family. Therefore, many results on the stability of the \( Q_1-P_0 \) element can be applied to \( P_1-P_0 \). For example, the reduced inf-sup constant of the \( Q_1-P_0 \) element is \( Ch \) on the square meshes of the unit square, therefore the reduced inf-sup constant of the \( P_1-P_0 \) element on the crisscross meshes of the unit square is \( Ch \) too. There are many results on the stability and approximability of the \( Q_1-P_0 \) element (see [4, 5, 8, 11, 12, 13, 14, 15, 16, 17 and 21].)

If a mesh family is stable for the \( Q_1-P_0 \) element, then the crisscross-refinement of the mesh family is reduced-stable for \( P_1-P_0 \). Then we show that the \( P_1-P_0 \) element has optimal approximations for the velocity and pressure. There are a wide range of stable families for the quadrilateral element \( Q_1-P_0 \), see [24]. Therefore the numerical solutions of the element \( P_1-P_0 \) are of the optimal order on the crisscross-refinements of all these families. The performance of the \( P_1-P_0 \) element on some general mesh families are also discussed in this paper.

As a byproduct of our analysis, we explicitly display a basis of all divergence-free continuous piecewise linear polynomials defined on the crisscross meshes of the unit square. Each basis function has a small support formed by a few triangles. Moreover, we show that the space of these divergence-free functions has the optimal approximation property. The analysis is supported by a numerical test.

There are 6 sections in this paper. In Section 2, we discuss the stability relation between \( P_1-P_0 \) and \( Q_1-P_0 \). In Section 3, we analyze the approximation properties of the \( P_1-P_0 \) element on the crisscross meshes of the unit square. We study the performance of the element on a general mesh family in Section 4. In Section 5, we display a basis of divergence-free piecewise linear functions. Finally in Section 6, we report some numerical results on the reduced stability and approximability of the \( P_1-P_0 \) element. We also illustrate the efficiency of using the newly discovered divergence-free basis in computation by a simple example.

## 2 Stabilities of \( P_1-P_0 \) and \( Q_1-P_0 \) elements

In this section, we shall study the relationship between \( P_1-P_0 \) and \( Q_1-P_0 \) finite elements. The \( \Omega \) under consideration could be a general polygonal domain. We denote a quadrilateral partition of \( \Omega \) by \( Q_h \) and its corresponding crisscross-refinement by \( T_h \), which is obtained by dividing each quadrilateral in \( Q_h \) by its two diagonals.

For each triangle \( T \in T_h \), let \( h_T \) denote the diameter of \( T \) and \( \rho_T \) the diameter of the circle inscribed in \( T \). A family of triangulations of \( \Omega \) is said to be regular if there is a positive \( \theta \)
onto the left-lower corner). For any quadrilateral \( W \in \mathcal{Q}_h \) with the vertices \( w_1, w_2, w_3, \) and \( w_4 \) (counterclockwise), there exists exactly one bilinear mapping \( F_W \in Q_1(\mathcal{W})^2 \) that maps \( \mathcal{W} \) onto \( W \) such that \( F_W(\hat{w}_i) = w_i \). Here

\[
\hat{Q}_k(\mathcal{W}) = \left\{ \sum_{0 \leq i, j \leq k} a_{ij} \hat{x}^i \hat{y}^j \mid a_{ij} \in \mathbb{R} \right\}
\]

On the quadrilateral \( W \) we define finite element space

\[
Q_k(W) = \{ v \circ F_W^{-1}, \forall v \in \hat{Q}_k(\mathcal{W}) \},
\]

where \( k \geq 1 \). On \( \mathcal{Q}_h \), we define

\[
M_k^0(\mathcal{Q}_h) = \{ v \in H^1(\Omega) \mid v|_W \in Q_k(W), \forall W \in \mathcal{Q}_h \},
\]

\[
M_k^{-1}(\mathcal{Q}_h) = \{ v \in L^2(\Omega) \mid v|_W \in Q_k(W), \forall W \in \mathcal{Q}_h \}.
\]

The finite element \( Q_1-P_0 \) and its associated spaces are defined as

\[
\tilde{V}_h = M_k^0(\mathcal{Q}_h) \cap H^1(\Omega),
\]

\[
\tilde{P}_h = M_k^{-1}(\mathcal{Q}_h),
\]

\[
\tilde{N}_h = \left\{ q \in \tilde{P}_h \mid b(v, q) = 0, \forall v \in \tilde{V}_h \right\},
\]

\[
\tilde{M}_h = L^2\text{-orthogonal complement of } \tilde{N}_h \text{ in } \tilde{P}_h.
\]

Let \( P_k(T) \) denote the polynomials with degree less than or equal to \( k \) defined on each triangle \( T \in \mathcal{T}_h \). On \( \mathcal{T}_h \) we define

\[
M_k^0(\mathcal{T}_h) = \{ v \in H^1(\Omega) \mid v|_T \in P_k(T), \forall T \in \mathcal{T}_h \},
\]

\[
M_k^{-1}(\mathcal{T}_h) = \{ v \in L^2(\Omega) \mid v|_T \in P_k(T), \forall T \in \mathcal{T}_h \}.
\]

For the \( P^1-P_0 \) element, we use the notations \( V_h, P_h, N_h, M_h, \) as defined by

\[
V_h = M_k^0(\mathcal{T}_h) \cap H^1(\Omega),
\]

\[
P_h = M_k^{-1}(\mathcal{T}_h),
\]

\[
N_h = \left\{ q \in P_h \mid b(v, q) = 0, \forall v \in V_h \right\},
\]

\[
M_h = L^2\text{-orthogonal complement of } N_h \text{ in } P_h.
\]

It is obvious that \( \tilde{P}_h \subset P_h \).

We call a polygonal region \( U \) of \( \mathcal{T}_h \) (resp. \( \mathcal{Q}_h \)) a macroelement if it is formed by some triangles of \( \mathcal{T}_h \) (resp. quadrilaterals of \( \mathcal{Q}_h \)). For a macroelement \( U \) of \( \mathcal{T}_h \) (resp. \( \mathcal{Q}_h \)), we define localizations \( V_h^U, P_h^U, N_h^U, \) and \( M_h^U \) (resp. \( \tilde{V}_h^U, \tilde{P}_h^U, \tilde{N}_h^U, \) and \( \tilde{M}_h^U \)) to \( U \) of the finite element spaces \( V_h, P_h, N_h, \) and \( M_h \) (resp. \( \tilde{V}_h, \tilde{P}_h, \tilde{N}_h, \) and \( \tilde{M}_h \)) by just replacing \( \Omega \) by \( U \) in (3) (resp. (4)).

\[
\frac{\rho_T}{r_T} \geq \theta. \tag{2}
\]
Lemma 2.1 Let $Q_h$ be a quadrilateral partition of a polygonal domain $\Omega$. If $T_h$ is the corresponding crisscross-refinement for $Q_h$, then

$$\tilde{N}_h \subset N_h.$$  

**Proof** We note a simple fact first. For any $v \in V_h$, there exists a $w \in \tilde{V}_h$ such that

$$b(v, q) = b(w, q), \forall q \in \tilde{P}_h. \quad (5)$$

Similarly, for any $w \in \tilde{V}_h$, there is $v \in V_h$ such that (5) holds.

Since any function $w \in \tilde{V}_h$ is linear on all the edges of $Q_h$, there exists a function $v \in V_h$ such that $v - w$ vanishes on all the edges of $Q_h$. Therefore, (5) follows by using the Green’s formula.

From (5), the lemma follows.

In order to study the structure of $N_h$, we consider $Q_h$ as a macroelement partition of $T_h$. For any quadrilateral $U \in Q_h$, there is a one-dimensional space of spurious pressure modes, with their support in $U$, associated with the singular vertex of $U$ (the intersection of the two diagonals of $U$, see Figure 1). For convenience, we denote the spurious pressure mode, shown in the second picture of Figure 1, by $\delta_U$ (values of $\delta_U$ are shown in Figure 1, where $a$, $b$, $c$, and $d$ are the lengths of the four interior edges). A detailed calculation of $\delta_U$ can be found in [19].

![Figure 1. Macroelement $U$ and spurious pressure mode $\delta_U$.](image)

**Lemma 2.2** Let $T_h$ be the crisscross-refinement of $Q_h$, then

$$N_h = \tilde{N}_h + \text{span} \{ \delta_U \mid \text{for all quadrilaterals } U \in Q_h \}.$$  

**Proof** Clearly, $\delta_U \in N_h$ for any $U \in Q_h$. Since $N_h(U)$ contains only linear combinations of $\chi_U$ (the characteristic function of $U$) and $\delta_U$ for each $U \in Q_h$, any function $q \in N_h$ must be in $\tilde{P}_h$ provided $q$ is orthogonal to all $\delta_U$’s. By (5), $q \in \tilde{N}_h$. Hence, the lemma follows.

**Corollary 2.1**

$$M_h \cap \tilde{P}_h = \tilde{M}_h.$$  

**Theorem 2.1** For a polygonal domain $\Omega$, let $Q_h$ be a quadrilateral partition of $\Omega$ and $T_h$ be the corresponding crisscross-refinement. Assume $T_h$ is regular. Then

$$C^{-1} \gamma_h(\tilde{V}_h, \tilde{M}_h) \leq \gamma_h(V_h, M_h) \leq C \gamma_h(\tilde{V}_h, \tilde{M}_h),$$

where $C$ is independent of $h$.  

4
Proof Let $\mathcal{Q}_h$ be a macroelement partition of $\mathcal{T}_h$. On each macroelement $U \in \mathcal{Q}_h$, we have
\[
\dim \mathbf{V}_h^U = 2, \quad \dim \mathbf{P}_h^U = 4, \\
\dim \mathbf{M}_h^U = 2, \quad \dim \mathbf{N}_h^U = 2.
\]
Following the arguments of Theorem 4.3.1 in [19], we can bound the local inf-sup constants $\gamma_h(\mathbf{V}_h^U, \mathbf{M}_h^U)$ on all macroelements $U$ in $\mathcal{Q}_h$ by a positive number which is independent of $h$. Applying the macroelement partition theorem (Theorem 3.2.1 in [19]), we know that the reduced inf-sup constant $\gamma_h(\mathbf{V}_h, \mathbf{M}_h)$ is determined by the stability of $\mathbf{V}_h \times (\mathbf{M}_h \cap \sum_{U \in \mathcal{Q}_h} \mathbf{N}_h^U) = \mathbf{V}_h \times \mathbf{\tilde{M}}_h$ (by the corollary). This implies that $\mathbf{V}_h \times \mathbf{M}_h$ has exactly the same stability as $\mathbf{V}_h \times \mathbf{\tilde{M}}_h$. If it is shown that $\mathbf{V}_h \times \mathbf{\tilde{M}}_h$ and $\mathbf{\tilde{V}}_h \times \mathbf{\tilde{M}}_h$ have the same stability, then the theorem is proven.

We first show that
\[
\gamma_h(\mathbf{V}_h, \mathbf{M}_h) \leq C^{-1} \gamma_h(\mathbf{\tilde{V}}_h, \mathbf{\tilde{M}}_h). \tag{6}
\]
For any $q \in \mathbf{\tilde{M}}_h$, there exists a function $w \in \mathbf{\tilde{V}}_h$ such that
\[
b(w, q) = \|q\|_{0,\Omega}^2, \\
\|w\|_{1,\Omega} \leq \frac{C}{\gamma_h(\mathbf{\tilde{V}}_h, \mathbf{\tilde{M}}_h)} \|q\|_{0,\Omega}. \tag{7}
\]
If we can construct a function $v \in \mathbf{V}_h$ such that
\[
b(v, q) = \|q\|_{0,\Omega}^2, \\
\|v\|_{1,\Omega} \leq \frac{C}{\gamma_h(\mathbf{V}_h, \mathbf{M}_h)} \|q\|_{0,\Omega},
\]
then (6) is proved.

Let $I_h := (I_h, I_h) : \mathbf{\tilde{V}}_h \to \mathbf{V}_h$ be the interpolation operator such that $I_h g$ and $g$ agree at every vertex in $\mathcal{T}_h$ for any $g \in \mathbf{\tilde{V}}_h$. For any function $g \in \mathbf{\tilde{V}}_h$, the interpolation error $g - I_h g$ vanishes at all the edges of the quadrilateral partition $\mathcal{Q}_h$. We will show that there is a constant $C$ independent of $h$ such that
\[
\|g - I_h g\|_{1,\Omega} \leq C \|g\|_{1,\Omega}, \tag{8}
\]
for any $g \in \mathbf{\tilde{V}}_h$. If this is the case, by taking $v = I_h w$, then
\[
\|v\|_{1,\Omega} \leq \|w - v\|_{1,\Omega} + \|w\|_{1,\Omega} \leq C \|w\|_{1,\Omega}.
\]
According to the arguments in the proof of Lemma 2.1 and (7), we have
\[
b(v, q) = b(w, q) = \|q\|_{0,\Omega}^2, \\
\|v\|_{1,\Omega} \leq \frac{C}{\gamma_h(\mathbf{V}_h, \mathbf{M}_h)} \|q\|_{0,\Omega}. \tag{10}
\]
This proves (6) with the assumption (8).

Let $U \in \mathcal{Q}_h$ be a macroelement. If we need to show that
\[
\|g - I_h g\|_{0,U} \leq C \|g\|_{0,U}, \tag{9}
\]
\[
|g - I_h g|_{1,U} \leq C |g|_{1,U}, \tag{10}
\]
for any $g \in Q_1(U)$, any $U \in Q_h$, and any $h > 0$ with constant $C$ independent of $h$. Let $E_0(\hat{U})$ denote the set of all the equivalence macroelements, see [1, 19 and 24], of the unit square $\hat{U}$ satisfying the regularity condition (2). Obviously, translations and dilations of $U$ do not affect (9) and (10). Therefore, for simplicity, we assume that the length of the longest diagonals of each macroelement in $E_0(\hat{U})$ is one unit, and that the intersection of the two diagonals of any macroelement in $E_0(\hat{U})$ has coordinates $(0, 0)$. For any macroelement $U$ in $E_0(\hat{U})$, we denote the intersection of its two diagonals by $v_5$ and the other four vertices by $v_1$, $v_2$, $v_3$, and $v_4$, clockwise. Hence, $S = \{(v_1, v_2, v_3, v_4, v_5) \mid U \in E_0(\hat{U})\}$ is a closed set in $\mathbb{R}^{10}$. Let $g \in Q_1(U)$ be arbitrary and it values at $v_i$ be $g_i$ for $i = 1, 2, 3, 4$. For convenience, we denote the vector $(g_1, g_2, g_3, g_4)^T$ by $\tilde{g}$. Since $(I_h g)(v_i) = g_i$ for $i = 1, 2, 3, 4$ and $(I_h g)(v_5)$ is determined by $g_1, g_2, g_3,$ and $g_4$, we have

$$\|g - I_h g\|_{0,U}^2 = g^1 A_U \tilde{g},$$

$$\|g\|_{0,U}^2 = \tilde{g}^T B_U \tilde{g},$$

$$|g - I_h g|^2_{1,U} = \tilde{g}^T C_U \tilde{g},$$

$$|g|_{1,U}^2 = \tilde{g}^T D_U \tilde{g}.$$  

Here $B_U$ is a symmetric positive definite and $A_U$, $C_U$, and $D_U$ are symmetric positive semidefinite. Clearly, the entries of $A_U$, $B_U$, $C_U$, and $D_U$ are continuous functions of $(v_1, v_2, v_3, v_4, v_5)$.

Since $S$ is a bounded closed set in $\mathbb{R}^{10}$, there exists a constant $C_1$ independent of $h$ such that

$$\|g - I_h g\|_{0,U} \leq C_1 |g|_{0,U}, \quad (11)$$

for any $U \in E_0(\hat{U})$.

It is easy to see that only the constant functions make both sides of (10) zero. Hence, the matrix $D_U$ always has exactly three positive eigenvalues which depend on $(v_1, v_2, v_3, v_4, v_5)$ continuously. Since $S$ is a bounded closed set in $\mathbb{R}^{10}$, the smallest nonzero eigenvalue of $D_U$ is bounded away from zero by a positive number independent of $h$. Due to the same reason, the largest eigenvalue of $C_U$ is bounded above by a constant independent of $h$. Therefore, there exists a constant $C_2$ independent of $h$ such that

$$|g - I_h g|_{1,U} \leq C_2 |g|_{1,U}, \quad (12)$$

for any $U \in E_0(\hat{U})$.

Combining (11) and (12), we obtain (9) and (10). Therefore, we have proven (6).

Using similar arguments, we can show that $\tilde{\gamma}_h \geq C \gamma_h$.

**Theorem 2.2** Let $Q_h$ be a quadrilateral partition of a polygonal domain $\Omega$ and $T_h$ be the corresponding crisscross-refinement. If $V_h \times P_h$ is stable, then $V_h \times M_h$ is stable. Moreover, if $(u_h, p_h) \in V_h \times M_h$ and $(u_h, p_h) \in V_h \times P_h$ solve (1), then

$$\|u - u_h\|_{1,\Omega} \leq C h \|u\|_{2,\Omega},$$

$$\|p - p_h\|_{0,\Omega} \leq C h (\|u\|_{2,\Omega} + \|p\|_{1,\Omega}). \quad (13)$$

Here the constant $C$ is independent of $h$, $p_h = p_h/N_h$, and we assume that $(u, p) \in H^2(\Omega) \times H^1(\Omega)$ solves the Stokes equations on the continuous level.

**Proof** Stability of $V_h \times M_h$ is a direct consequence of Theorem 2.1. It is known that $M_0^{-1}(Q_h)/R \subset M_h$ implies that the approximation properties of $M_h$ are as good as those of $P_h$. Therefore, by the stability theorem established in [7], (13) follows (the error estimate of velocity is decoupled from $\|p\|_{1,\Omega}$ because of $\text{div} u_h = 0$).
For the $Q_1-P_0$ element, one stable mesh family was identified in [24]. On the crisscross-refinements of all these stable families, the $P_1-P_0$ element is reduced-stable and the numerical solution is optimal.

The pressure $\bar{p}_h \in M_h$ can be recovered from $p_h \in P_h$ by applying a simple postprocess quadrilateral by quadrilateral. That is, on each quadrilateral $U \in \mathcal{Q}_h$,

$$\bar{p}_h|_U = p_h|_U - \frac{\int_U p_h \delta_U}{\int_U \delta^2_U} \delta_U.$$

As a direct application of Lemma 2.2, Theorem 2.1, and the stability results of the element $Q_1-P_0$ in [4, 20, and 17], we get the following theorem.

**Theorem 2.3** Let $\mathcal{Q}_h$, $h = 1/n$, be a square mesh of the unit square and $\mathcal{T}_h$ be the corresponding crisscross-refinement. Then

$$\gamma_h(\mathcal{V}_h, M_h) = Ch, \quad \dim N_h = n^2 + 2.$$

However, the approximation properties of the numerical solution of the $P_1-P_0$ element on the crisscross-refinement (the mesh is also called crisscross mesh) of the square mesh of the unit square can not be answered directly by applying the stability theorem established in [7] since the reduced inf-sup constant is $Ch$. This will be answered in the next section.

### 3 Approximation properties on crisscross meshes

In this section, we show that the numerical solutions of (1) converge to the true solution of the Stokes equations (with Dirichlet boundary condition) at a rate of $h$ on the crisscross meshes of the unit square. Moreover, we will display a way to recover the numerical solution for the pressure. Since $\mathcal{V}_h \times M_h$ is unstable, this means that $M_h$ is still too large. Therefore, we need to remove more modes from $M_h$ while trying to preserve the approximation properties for the remaining pressure space $S_h$. It is then necessary to define a new velocity space $W_h \subset \mathcal{V}_h$ such that not only is $W_h \times S_h$ stable but also the velocity from $W_h \times S_h$ is exactly the same as $u_h$ from $\mathcal{V}_h \times M_h$. Of course, the space $W_h$ should possess good approximation properties.

The crucial matter here is to determine the bad modes in $M_h$. Let $h = 1/n$ and $n = 4k$ for some positive integer $k$, and let $\mathcal{T}_h$ be the crisscross mesh of the unit square. For convenience, we denote $(\alpha, \beta)$ the vertex in $\mathcal{T}_h$ with coordinates $(\alpha h, \beta h)$, and $\delta_{\alpha+1/2,\beta+1/2}$ the spurious pressure mode associated with the singular vertex $(\alpha, \beta)$ (see Figure 2). Let $\mathcal{Q}_{4h}$ be a macroelement partition of $\mathcal{T}_h$ such that every $U \in \mathcal{Q}_{4h}$ has $16 h \times h$ squares. Therefore, $U$ consists of 64 triangles. From Lemma 2.2, we know that $N^U_h$ is the space spanned by the functions $\chi^U$ (the characteristic function with support $U$), $\delta_U$ (the checkerboard mode with support $U$, shown in Figure 2), and $\delta_{i+1/2, j+1/2}$, where $(i + 1/2, j + 1/2)$ is a singular vertex in $U$. Obviously, the $\delta_{i+1/2, j+1/2}$’s are not in $M_h$. Since the global checkerboard mode $\delta_{\Omega} \in \text{span}\{\delta_U, \forall U \in \mathcal{Q}_{4h}\}$, we need to remove $\text{span}\{\delta_U, \forall U \in \mathcal{Q}_{4h}\}$ from $M_h$. 

7
Figure 2. $\delta_{\alpha+1/2,\beta+1/2}$ and $\delta_U$.

Define

\[
\bar{N}_h = \text{span}\{1, \delta_{i+\frac{1}{2}, j+\frac{1}{2}}, \delta_U, 1 \leq i, j \leq n-1, \forall U \in Q_{4h}\},
\]

\[
S_h = L^2 \text{ orthogonal complement of } \bar{N}_h \text{ in } P_h,
\]

\[
W_h = \{ v \in V_h \mid b(v, q) = 0, \forall q \in \bar{N}_h \},
\]

\[
S^U_h = \chi^U S_h,
\]

\[
W^U_h = \{ v \in W_h \mid \text{support} v \subset U \}.
\]

**Theorem 3.1** Let $\mathcal{T}_h$ be a crisscross mesh of the unit square. Assume $(u, p) \in \tilde{H}^2(\Omega) \times H^1(\Omega)$ is the true solution of the Stokes equations and $(\tilde{u}_h, \tilde{p}_h) \in W_h \times S_h$ solves (1), then

\[
\|u - \tilde{u}_h\|_{1, \Omega} \leq Ch\|u\|_{2, \Omega},
\]

\[
\|p - \tilde{p}_h\|_{0, \Omega} \leq Ch(\|u\|_{2, \Omega} + \|p\|_{1, \Omega}).
\]

**Proof** It is easy to verify that

\[
N_h \subset \bar{N}_h, \quad M^0_1(\mathcal{T}_h) \cap H^1(\Omega) \subset W_h, \quad \text{and} \quad M^{-1}_0(Q_{4h}) \subset S_h.
\]

Therefore, $W_h \times S_h$ has good approximation properties. It only remains to prove that $W_h \times S_h$ is stable.

Define

\[
\bar{N}^U_h = \{ q \in S^U_h \mid b(v, q) = 0, \forall v \in W^U_h \}.
\]

Then Lemma 2.2 implies that $\bar{N}^U_h$ contains only constant functions. Therefore, $W_h \times (S_h \cap (\bigcup_{U \in Q_{4h}} \bar{N}^U_h))$ is stable by the fact that $[M^0_1(\mathcal{T}_h) \cap H^1(\Omega)] \times M^{-1}_0(Q_{4h})$ is stable. Finally, by using the macroelement partition theorem, see [1, 19 and 24], we have that $W_h \times S_h$ is stable.

**Theorem 3.2** Let $\mathcal{T}_h$ be the crisscross mesh of the unit square with $h = 1/(4k)$ for some positive integer $k$. If $(u, p) \in \tilde{H}^2(\Omega) \times H^1(\Omega)$ is the true solution of the Stokes equations, then the solution $(u_h, p_h) \in V_h \times P_h$ of (1) satisfies

\[
\|u - u_h\|_{1, \Omega} \leq Ch\|u\|_{2, \Omega},
\]

\[
p_h = \tilde{p}_h + \bar{N}_h,
\]

and the pressure can be recovered from $p_h$ by a postprocess.

**Proof** Clearly, the solution $(u_h, p_h) \in V_h \times P_h$ of

\[
\begin{align*}
& a(u_h, v) + b(v, p_h) = (f, v), \quad \forall v \in V_h, \\
& b(u_h, q) = 0, \quad \forall q \in P_h
\end{align*}
\]
satisfies
\[ a(u_h, v) + b(v, p_h) = (f, v), \quad \forall v \in W_h, \]
\[ b(u_h, q) = 0, \quad \forall q \in S_h. \tag{14} \]

Since \( u_h \in W_h \) and \( p_h = (p_h/\tilde{N}_h) + n_h \) for some \( n_h \in \tilde{N}_h \), (14) implies
\[ a(u_h, v) + b(v, p_h/\tilde{N}_h) = (f, v), \quad \forall v \in W_h, \]
\[ b(u_h, q) = 0, \quad \forall q \in S_h. \]

Since the solution of (1) in \( W_h \times S_h \) is unique, we get
\[ u_h = \tilde{u}_h \quad \text{and} \quad p_h/\tilde{N}_h = \tilde{p}_h. \]

By Theorem 3.1, the velocity \( u_h \) and recovered pressure \( \tilde{p}_h \) have optimal rates of convergence. It is simple to recover \( \tilde{p}_h \) from \( p_h \) since \( \tilde{N}_h \) is known.

4 Approximation properties on a general mesh family

The \( \mathcal{P}^1-\mathcal{P}^0 \) element is further analyzed on a general mesh family in this section. On this family of meshes, it is shown that the finite element solution for the velocity converges at an order \( h \) and the pressure can be recovered by a simple postprocess.

The triangulation \( T_h \) is formed in the following way. First, we partition the polygonal domain \( \Omega \) into quadrilaterals. This quadrilateral partition is denoted by \( Q_{4h} \). Secondly, each quadrilateral in \( Q_{4h} \) is divided into four sub-quadrilaterals by linking the intersection of its two diagonals to the middle point of each edge, so \( Q_{2h} \) is formed. Repeating the above process to all quadrilaterals in \( Q_{2h} \), we have \( Q_h \). Finally, partitioning each quadrilateral in \( Q_h \) into 4 triangles by its two diagonals, we obtain the triangulation \( T_h \). The first figure in Figure 3 shows how to partition a quadrilateral in \( Q_{4h} \) into 4 quadrilaterals in \( Q_{2h} \). The second figure shows a macroelement in \( Q_{4h} \) with 64 triangles in \( T_h \).

We will show that on the triangulation \( T_h \), the numerical solution \( u_h \) has an optimal rate of convergence and a pressure with an optimal rate of convergence can be recovered from \( p_h \). The measure to achieve this objective is quite similar to what we used for the crisscross mesh. We first remove all possible “bad modes” from \( P_h \), and then consider the pressure in the remaining pressure space. The key issue is to determine bad pressure modes in \( P_h \). We know that spurious pressure modes associated with each singular vertex must be removed from the pressure space \( P_h \). However, this is not enough to guarantee the stability according to our experience with the crisscross mesh. Since the crisscross mesh is a special case of \( T_h \), we definitely need to remove those pressure modes which may degenerate to local spurious modes. Based on this consideration, we will remove all multiples of the mode (shown in Figure 4) on each macroelement in \( Q_{2h} \) from the pressure space \( P_h \).
structure, we can show that we need to show that \( U \) is stable. Therefore, \( W \) is stable. Therefore, by using the macroelement partition theorem we can prove that \( W \) is stable by the fact that \( \delta_{U_{2h}} \) denote the spurious pressure mode associated with the singular vertex in \( U_h \in Q_h \). We can easily conclude that

\[
\int_T \delta_{U_{2h}} = 0,
\]

for any \( T \in \mathcal{T}_{2h} \), any \( U_{2h} \in Q_{2h} \).

Define

\[
N_h = \text{span}\{1, \delta_{U_h}, \delta_{U_{2h}}, \forall U_h \in Q_h, \forall U_{2h} \in Q_{2h}\},
\]

\[
S_h = L^2 \text{ orthogonal complement of } N_h \text{ in } P_h,
\]

\[
W_h = \{ v \in V_h \mid b(v, q) = 0, \forall q \in N_h \},
\]

\[
S_{h,2h} = \chi_{U_{2h}} S_h,
\]

\[
W_{h,2h} = \{ v \in W_h \mid \text{ support } v \subset U_{2h} \}.
\]

**Theorem 4.1** Let \( \mathcal{T}_h \) be a regular triangulation defined above. If \( (u, p) \in H^2(\Omega) \times H^1(\Omega) \) is the true solution of the Stokes equations and \( (\tilde{u}_h, \tilde{p}_h) \in W_h \times S_h \) solves (1), then

\[
\|u - \tilde{u}_h\|_{1,\Omega} \leq C h \|u\|_{2,\Omega},
\]

\[
\|p - \tilde{p}_h\|_{0,\Omega} \leq C h (\|u\|_{2,\Omega} + \|p\|_{1,\Omega}).
\]

**Proof** It is expected that \( N_h \subset \tilde{N}_h \), if \( \mathcal{T}_h \) is a crisscross mesh. Since \( \mathcal{T}_{h,2h} \) has the special structure, we can show that

\[
M_0^{-1}(Q_{2h}) \subset S_h \quad \text{and} \quad [M_0^{-1}(\mathcal{T}_{2h}) \cap \bar{H}^1(\Omega)] \subset W_h.
\]

Therefore, \( W_h \times S_h \) has good approximation properties. If we can prove that \( W_h \times S_h \) is stable, then the proof is done.

Define

\[
\tilde{N}_{h}^{U_{4h}} = \{ q \in S_{h,2h} \mid b(v, q) = 0, \forall v \in W_{h,2h} \}.
\]

We need to show that \( \tilde{N}_{h}^{U_{4h}} \) contains only constant functions. If it does for any \( U_{4h} \in Q_{4h} \), then \( W_h \times (S_h \cap (\cup_{U_{4h} \in Q_{4h}} \tilde{N}_{h}^{U_{4h}})) \) is stable by the fact that \( [M_0^{-1}(\mathcal{T}_{2h}) \cap \bar{H}^1(\Omega)] \times M_0^{-1}(Q_{4h}) \) is stable. Therefore, by using the macroelement partition theorem we can prove that \( W_h \times S_h \) is stable.

We show that \( \dim \tilde{N}_{h}^{U_{4h}} = 1 \). Since \( \delta_{U_h} \in N_h \), if \( q \in \tilde{N}_{h}^{U_{4h}} \), then \( q \) is a constant on each \( U_h \in U_{4h} \). Hence, \( q \) must be a constant on each \( U_{2h} \in U_{4h} \) since we already removed \( \delta_{U_{2h}} \) from the pressure space. Therefore, \( q \) must be a constant on each of the four \( U_{2h} \in U_{4h} \). A simple computation shows that \( q \) must be a constant on \( U_{4h} \).
Theorem 4.2 Let $\mathcal{T}_h$ be the triangulation defined at the beginning of this section, from a specially constructed $\mathcal{Q}_h$, then the numerical solution $(u_h, p_h) \in V_h \times P_h$ satisfies

$$
\|u - u_h\|_{1, \Omega} \leq Ch\|u\|_{2, \Omega},
$$

and $p_h/\bar{N}_h = \tilde{p}_h$. Here we assume $(u, p) \in \bar{H}^2(\Omega) \times H^1(\Omega)$ solves the Stokes equations.

Proof Use similar arguments as in the proof of Theorem 3.2.

5 A basis of divergence-free piecewise linear functions

In this section, we display a basis for $Z_h$, the space of all divergence-free continuous piecewise linear polynomials on the crisscross mesh $\mathcal{T}_h$. All the basis functions have a very small local support and the space $Z_h$ has optimal approximation properties.

Let the domain $\Omega$ be the unit square and $\mathcal{Q}_h$, $h = 1/n$ a partition of $\Omega$ which contains $n \times n$ equal small squares. The triangulation $\mathcal{T}_h$ is the crisscross-refinement of $\mathcal{Q}_h$.

Define

$$
V_h = M_0^1(\mathcal{T}_h) \cap \bar{H}^1(\Omega),
$$

$$
Z_h = \{ v \in V_h \mid \text{div } v = 0 \}.
$$

A simple calculation shows that

$$
\dim V_h = 4n^2 - 4n + 2.
$$

In order to study $Z_h$, we define a “pressure” space $P_h$ as

$$
P_h = M_0^{-1}(\mathcal{T}_h).
$$

The analysis of the properties of $Z_h$ can be carried out using the frame work of the analysis of the $P^1-P^0$ finite element for the Stokes equations with Dirichlet boundary conditions. Since $\text{div } V_h \subset P_h$, we have

$$
Z_h = \{ v \in V_h \mid \text{div } v = 0 \} = \{ v \in V_h \|b(v, q) = 0, \forall q \in P_h\}.
$$

Lemma 5.1 On the crisscross triangulation $\mathcal{T}_h$ with $h = 1/n$,

$$
\dim Z_h = (n - 2)^2.
$$

Proof Since $\dim P_h = 4n^2$ and $\dim N_h = n^2 + 2$ (see Theorem 2.3), the lemma holds.

In order to find a basis for $Z_h$, we first consider a small macroelement $U$ with size $3h \times 3h$ (see Figure 5).

![Figure 5. Macroelement U.](image-url)
We denote the interior vertices of $U$ by $1, 2, \ldots, 13$, as shown in Figure 5, and denote the nodal basis functions by $\phi_1, \phi_2, \cdots, \phi_{13}$ accordingly. Let $V_h^U$ denote the subspace of $V_h$ such that all the functions in $V_h^U$ have supports contained in $U$. We look for functions $v = \Sigma_{i=1}^{13}(u_i, v_i)\phi_i \in V_h^U$ such that $\text{div} \, v = 0$ in $U$. Namely, we need to solve a system of 36 linear equations in 26 unknowns. Since $\dim Z_h^U = 1$, we know that this system has a one-dimensional solution space. After some algebraic computations, we find that the solution space is

$$Z_h^U = \text{span}\{z^U\},$$

where $z^U = (\xi, \eta)$ and

$$\xi = \phi_2 - \phi_8 + \phi_{10} + \phi_{11} - \phi_{12} - \phi_{13},$$

$$\eta = -\phi_4 + \phi_6 - \phi_{10} + \phi_{11} - \phi_{12} + \phi_{13}.$$ 

See Figure 6 for graphs of $\xi$ and $\eta$. Since there are exactly $(n - 2)^2$ different macroelement with size $3h \times 3h$ in $T_h$ — the set of all these macroelements is named by $U_h$—we find a basis for $Z_h$. By the results of Section 4, we have

**Theorem 5.1** Let $T_h$ be the crisscross mesh of the unit square with $h = 1/n$. Then

$$\{z^U \mid \forall U \in U_h\},$$

form a basis of $Z_h$. Furthermore, if $u \in \tilde{H}^2(\Omega)$ and $\text{div} \, u = 0$, then

$$\inf_{v \in Z_h} \|u - v\|_{1, \Omega} \leq Ch\|u\|_{2, \Omega}.$$ 

Figure 6. The shape of the two components of $z^U$.

6 Numerical tests

In this section, we report some results of numerical experiments. First we calculate the reduced inf-sup constant of the element $P_1-P_0$ on a reduced-stable family, which is a crisscross-refinement of a stable quadrilateral family for the $Q_1-P_0$ element, see [24]. We consider problem (1) when $\Omega$ is the unit square. The unit square is first partitioned into $n \times n$ small squares, $2h = 1/n$. The partition is denoted by $Q_{2h}$. The $Q_h$ is then defined by partitioning each square of $Q_{2h}$ into 5 quadrilaterals(see Figure 7). Finally, the triangulation $T_h$ is the crisscross-refinement of $Q_h$ as depicted in Figure 7.
By the work [24], we know the $Q_1-P_0$ element is stable on $Q_h$. Therefore, $P_1-P_0$ is reduced-stable on $T_h$, and $\dim N_h = 5n^2 + 1$ (Lemma 2.2). Although there is a relatively large space of spurious pressure modes involved, all these modes can be filter out from the numerical approximation $p_h$ quite easily, quadrilateral by quadrilateral. Of course, the velocity approximation $u_h$ is of the optimal order, and no recovery is needed.

The reduced inf-sup constant $\gamma_h(V_h, N_h)$ and $\dim N_h$ for $1/h = 2n = 2, 4, 6, \ldots, 16$ are reported in Table 1. The reduced inf-sup constant is clearly bounded below, and the bound is a relatively large number.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\gamma_h(V_h, M_h)$</th>
<th>$\dim N_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/2$</td>
<td>0.31808</td>
<td>6</td>
</tr>
<tr>
<td>$1/4$</td>
<td>0.31314</td>
<td>21</td>
</tr>
<tr>
<td>$1/6$</td>
<td>0.31378</td>
<td>46</td>
</tr>
<tr>
<td>$1/8$</td>
<td>0.31481</td>
<td>81</td>
</tr>
<tr>
<td>$1/10$</td>
<td>0.31615</td>
<td>126</td>
</tr>
<tr>
<td>$1/13$</td>
<td>0.31653</td>
<td>181</td>
</tr>
<tr>
<td>$1/14$</td>
<td>0.31660</td>
<td>246</td>
</tr>
<tr>
<td>$1/16$</td>
<td>0.31693</td>
<td>321</td>
</tr>
</tbody>
</table>

Finally, we use the newly discovered div-free $P_1$ elements on the criss-cross grids (cf. Figure 5) to solve the following model Stokes equations:

\[
a(u, v) - b(v, p) = (f, v), \quad \forall v \in V, \\
b(u_h, q) = 0, \quad \forall q \in P,
\]

defined on the unit square $\Omega = [0, 1]^2$, where

\[
f = -\Delta \text{curl} g + \nabla g_{xx} = \begin{pmatrix} -g_{yxx} - g_{yyy} - g_{xxx} \\ g_{xxx} + g_{xxy} - g_{yxx} \end{pmatrix}.
\]

(15)

with $g = 64(x - x^2)^2(y - y^2)^2$. The exact solution is

\[ u = \text{curl} g. \]

We depict the first component of $u$ ($u_h$ in fact) in Figure 8.
Thanks to the newly discovered div-free basis, the problem (15) is reduced to, find $u_h \in Z_h$ such that

$$a(u_h, v) = (f, v), \quad \forall v \in Z_h,$$

(16)

where $Z_h$ is the div-free $P_1$ finite element space defined in Section 5. We note that, comparing to the traditional non-positive definite finite element systems, we have a positive definite matrix for the discrete linear system (16). Further, the number of unknowns in the finite element equations (16) is much less than one-eighth of that in the standard mixed finite element equations, see the fourth and the fifth columns of Table 2. From the fifth column of Table 2, it is apparent that the div-free finite element solution converges at the optimal order, $O(h^2)$. At the end, in Figure 9, we plot one component of error $u - u_h$ on the grid level 4. We can see that the nodal error is much larger at the criss-cross points (the center of each square), compared with that at the square vertices.

<table>
<thead>
<tr>
<th>Level</th>
<th>$h$</th>
<th># elements</th>
<th>$\text{dim } V_h + \text{dim } P_h$</th>
<th>$\text{dim } Z_h$</th>
<th>$| u - u_h |_{L^\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1/4</td>
<td>64</td>
<td>146</td>
<td>4</td>
<td>0.21398925</td>
</tr>
<tr>
<td>4</td>
<td>1/8</td>
<td>256</td>
<td>546</td>
<td>36</td>
<td>0.07262499</td>
</tr>
<tr>
<td>5</td>
<td>1/16</td>
<td>1024</td>
<td>2114</td>
<td>196</td>
<td>0.02063174</td>
</tr>
<tr>
<td>6</td>
<td>1/32</td>
<td>4096</td>
<td>8322</td>
<td>900</td>
<td>0.00549276</td>
</tr>
</tbody>
</table>

Acknowledgement The author would like to thank Professor Douglas N. Arnold for his encouragement, advice, and support during the years of the author’s study at the Pennsylvania
State University.

References.


