Analytical and numerical solutions for torsional flow between coaxial discs with heat transfer

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Abstract

We consider non-isothermal torsional flow between two coaxial parallel plates with heat transfer at the upper rotating plate, constant temperature on the lower stationary plate, and no heat loss at the fluid-air interface. Viscous heating is modelled by a Nahme law with exponential dependence on temperature. Due to the highly nonlinear nature of the governing equations an exact solution is not feasible. Therefore we solve the problem using both numerical and perturbations methods. Specifically, analytical solutions are obtained using asymptotic expansions based on the aspect ratio and the Nahme-Griffith number, a measure of viscous heating, as perturbation parameters. The numerical solutions are obtained by a finite element method. Good agreement is found between the analytical and numerical solutions in appropriate parameter range. In viscometric applications the torque exerted by the fluid on the lower plate is an important quantity. For isothermal flow the dimensionless torque can be easily calculated. In this paper we obtain an analytical formula that can be used to calculate non-isothermal correction to the torque.

Keywords. Parallel-plate flow, viscous heating, axisymmetric finite elements. inf-sup condition, singular vertex,


1 Introduction

The problem of viscous heating in viscometric flow of Newtonian and non-Newtonian liquids is of both theoretical and practical interest. An exact solution was obtained for flow of a Newtonian fluid between two infinite parallel plates by Nahme in 1940 [1]. The viscosity was modelled by an exponential function of temperature. A similar result was obtained by Kearsley [2] for the pressure gradient flow of a Newtonian fluid in a tube. Bird and Turian [3] analyzed the viscous heating problem for the flow of a Newtonian fluid between a cone and a plate. The equations for the temperature and velocity were uncoupled by assuming an isothermal velocity profile. The variational form for the temperature was then solved numerically. Isothermal boundary conditions were imposed at the physical boundaries and zero heat loss imposed at the air-liquid interface. Their analysis showed that viscous heating can lead to observable errors in the cone-and-plate viscometer. In a subsequent paper, Turian and Bird [4] extended the theoretical investigation to plane Couette flow of Newtonian fluids with temperature dependent viscosity and thermal conductivity. The thermal conductivity was assumed

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to be a linear function of temperature while the viscosity was assumed to obey a Nahme law with exponential dependence on temperature. A regular perturbation solution in powers of the Brinkman number was obtained for the velocity and temperature. A perturbation solution in powers of Brinkman number were later obtained for non-Newtonian liquids described by the power-law and Ellis’ models [5]. The boundary conditions were the same as those used in [4]. Exact analytical solutions have also been obtained by Martin [6] for flows between infinite concentric cylinders and infinite parallel plates for Newtonian and power law fluids. Two types of boundary conditions were considered, one in which both surfaces were isothermal and the other in which one was isothermal and the other adiabatic. Closed form solutions similar to those of Nahme and Kearsley were obtained for isothermal and adiabatic boundary conditions by Gavis and Laurence [7].

These early analytical studied were motivated by the need to quantify the deviation form isothermal flow in viscometers when viscous dissipation is significant and to provide simple formulas to correct for such deviations in quantities such as the torque on the stationary plate in parallel-plate flow. In recent years renewed interest in this matter has been spurred by experiments in elastic instabilities of viscoelastic fluids. It has been observed that viscous heating could lead to qualitatively and quantitatively significant deviation from isothermal theory in the stability property of a viscoelastic fluid. Experimental study of the stability of isothermal flow of viscoelastic torsional flow was first reported by Larson et al [8] and McKinley et al [9]. Linear stability analysis was first carried out by Oztekin and Brown [10] for flow between parallel-plate flow in which the boundary condition at the fluid-air interface was neglected. Linear stability results for flow in a finite domain incorporating boundary conditions at the free surface have also been considered [11, 12, 13].

Experiments on the effect of viscous heating on the stability of torsional flow of viscoelastic fluids was first reported by Rothstein and McKinley [14]. The results were found to be remarkably different from those of isothermal flows. It was shown that viscous heating tended to stabilize the flow. A linear stability for the non-isothermal problem was later analyzed by Olagunju et al [15] which agrees qualitatively with their experimental results. In that paper isothermal boundary conditions on the plates were used while those at the free surface were neglected. However, as noted by Arigo [16], it is practically impossible to control the temperature on upper rotating cone (or plate in case of parallel plate torsional viscometer). He also notes that the upper rotating cone (or plate) is cooled by convection of ambient air at 23-24 Celsius. A more realistic set of boundary conditions is to treat the bottom plate as isothermal, the free surface as an insulated boundary and the top plate as a thermal mass. The insulation boundary condition surface can be justified on the grounds that for a small gap thickness, the surface available for heat transfer to the ambience through the radial interface is practically negligible. It is hoped that this will give results that are in quantitative agreement with experiments. A heat transfer boundary condition was used for the viscoelastic Taylor-Couette problem by Al-Mubaiyedh et al [17]. In their study, the heat transfer boundary condition was used to numerically simulate the experiments of Baumert and Muller [18, 19]. Another assumption that was made in [15] is that the parallel plates are infinite in extent. In order to obtain better agreement between theory and experiments we think that it is necessary to relax these assumptions. As a first step in this direction we study the effect of the finite geometry and the more realistic boundary conditions on the base flow. In [20], Olagunju showed that for torsional flow of a viscoelastic fluid the base flow is not always purely circumferential. He showed that viscous heating leads to secondary flows with recirculating roll cells in the base solution.

In this paper, we obtain perturbation and numerical solutions for the flow between two
parallel plates of a Newtonian fluid with temperature dependent viscosity. Specifically we will assume an exponential dependence of the Nahme type. As noted above this problem has been solved for flow between two infinite parallel plates [1, 4]. In this limit the problem reduces to two coupled ordinary differential equations for the temperature and azimuthal velocity. We propose to solve the problem in a finite geometry with a fluid-air interface. In this case we obtain two coupled partial differential equations for the temperature and velocity. Bird and Turian [5] have also analyzed the problem in a finite geometry between a cone and a plate. However they assumed that the velocity profile was isothermal thereby reducing the problem to a single partial differential equation for the temperature. In all previous work that we know the boundary conditions are either isothermal on both plates or isothermal on one plate and zero heat transfer on the other. We will adopt the more realistic boundary conditions described above, namely isothermal condition on the stationary plate, heat transfer on the upper plate and zero heat transfer at fluid-air interface. To the best of our knowledge exact analytical solutions for this problem have not been previously reported. Having an analytical solution for this problem will enable one to estimate errors in the torque calculations due to heat transfer and edge effects if needed. This is important in viscometry. Analytical solutions can also be used to validate numerical calculations. For viscoelastic flows in which secondary flow in the base flow is weak or non-existent the solution provided here provides an accurate approximation to the base flow needed in any linear stability analysis.

2 Governing equations

We consider the flow of a fluid in the region between two coaxial parallel plates of radius $a$ and separation $h$ in which the top plate rotates at a constant angular speed $\omega$ and the bottom plate is stationary. Following Olagunju [21], the nondimensionalized equations governing the primary flow for the azimuthal velocity $W$ and a scaled temperature $\Theta$ are given in cylindrical coordinates as,

$$\frac{\partial^2 W}{\partial z^2} + \alpha^2 \left( \frac{\partial^2 W}{\partial r^2} + \frac{1}{r} \frac{\partial W}{\partial r} - \frac{W}{r^2} \right) = \frac{\partial \Theta}{\partial z} \frac{\partial W}{\partial z} + \alpha^2 \frac{\partial \Theta}{\partial r} \left( \frac{\partial W}{\partial r} - \frac{W}{r} \right) \tag{1}$$

$$\frac{\partial^2 \Theta}{\partial z^2} + \alpha^2 \left( \frac{\partial^2 \Theta}{\partial r^2} + \frac{1}{r} \frac{\partial \Theta}{\partial r} \right) = -Na_0 e^{-\Theta} \left[ \left( \frac{\partial W}{\partial z} \right)^2 + \alpha^2 \left( \frac{\partial W}{\partial r} - \frac{W}{r} \right)^2 \right] \tag{2}$$

with boundary conditions,

$$\text{at } z = 0, \quad W = 0, \quad \Theta = \vartheta_w \tag{3}$$

$$\text{at } z = 1, \quad W = r, \quad \frac{\partial \Theta}{\partial z} + B \Theta = B \vartheta_a \tag{4}$$

$$\text{at } r = 0, \quad W = 0, \quad |\Theta| < \infty \tag{5}$$

$$\text{at } r = 1, \quad \frac{\partial W}{\partial r} - \frac{W}{r} = 0, \quad \frac{\partial \Theta}{\partial r} = 0 \tag{6}$$

Here $\vartheta_w$ and $\vartheta_a$ are the scaled temperature at the stationary plate and the ambient. The aspect ratio $\alpha = h/a$ and the modified Biot number $B$ is defined in Appendix A.1. The Nahme-Griffith number $Na_0$ which is a measure of viscous heating in the fluid is zero for isothermal flows. It is defined as, $Na_0 \equiv (\eta_0 \delta a^2 \omega^2)/(kT_0)$. The quantity $\eta_0$ is the isothermal viscosity, $\delta$ is a thermal sensitivity parameter, $k$ the thermal conductivity and $T_0$ a reference
temperature [14]. Since the equations are nonlinear finding an exact analytical solution valid for all parameter values is impractical. Therefore we will solve the equations numerically using a finite element method. We will also obtain analytical solutions using perturbation expansions in Nahme-Griffith number \( Na \) and the aspect ration \( \alpha \).

3 Analytical solutions

3.1 An exact solution for \( \alpha = 0 \) and \( B = 0 \)

An exact solution of equations (1)-(6) can be found for \( \alpha = 0, B = 0 \) and all values of \( Na_0 \). This corresponds to plane Couette flow with the upper plate insulated [5]. This solution does not satisfy the boundary conditions at the fluid-air interface \( r = 0 \). However, we will show that it provides an excellent approximation to the solution for small \( \alpha \) except very close to \( r = 1 \). In addition we will show that the torque exerted by the fluid on the lower plate is very well approximated by this exact solution when the aspect ratio \( \alpha \) is small. An analytical formula is provided which can be used to calculate non-isothermal correction to the torque, as is often required in rheometry. Note that the exact solution obtained by Nahme [1] corresponds to \( \alpha = 0 \) and \( B = \infty \).

Setting \( \alpha \) and \( B \) to zero, equation (1) - (6) reduce to the following.

\[
\frac{d^2W}{dz^2} = \frac{d\Theta}{dz} \frac{dW}{dz} \tag{7}
\]

\[
\frac{d^2\Theta}{dz^2} = -Na_0e^{-\Theta} \left( \frac{d\omega}{dz} \right)^2 \tag{8}
\]

The boundary conditions are

\[
W = 0, \quad \Theta = \vartheta_w, \quad \text{for} \quad z = 0 \tag{9}
\]

and

\[
W = r, \quad \frac{\partial \Theta}{\partial z} = 0, \quad \text{for} \quad z = 1. \tag{10}
\]

It is straightforward to obtain the solution which is given by

\[
W = \frac{r}{2} - \frac{1}{\mu Na \tanh[E(1 - 2z)]}, \tag{11}
\]

\[
\Theta = \vartheta_w + \ln \left[ \left( 1 + \frac{r^2Na}{8} \right) \text{sech}^2[E(1 - 2z)] \right] \tag{12}
\]

where \( Na = Na_0e^{-\vartheta_w} \),

\[ E = \tanh^{-1}(r\mu Na/2) \]

and

\[ \mu = \left[ 2Na(1 + \frac{r^2}{8}Na) \right]^{-\frac{1}{2}}. \]

The dimensionless torque on the lower plate is defined

\[
T = \int_0^1 \int_0^1 \left( \frac{dW}{dz} \right)_{z=0} r^2 dr dz. \]
Using the solution obtained above, a series solution for $T$ valid for $Na < 2$ is given by

$$T = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \left( \frac{Na}{2} \right)^{n+k} \frac{(n+k)!}{n!k!} \frac{1}{(2n+1)(2n+2k+4)}$$

(13)

For other values of $Na$ the integral can easily be computed numerically. These solutions will be compared to asymptotic and numerical solutions below.

### 3.2 Asymptotic solution for $Na \ll 1$ : $\vartheta_w = \vartheta_a$

We seek a regular expansion in Nahme number for $W$ and $\Theta$ as follows.

$$W = W_0 + Na W_1 + O(Na^2), \quad \Theta = \Theta_0 + Na \Theta_1 + O(Na^2)$$

(14)

Here and in what follows $Na = Na_0 e^{-\vartheta_w}$. The governing equations for $W_0$ and $\Theta_0$ are,

$$\frac{\partial^2 W_0}{\partial z^2} + \alpha^2 \left( \frac{\partial^2 W_0}{\partial r^2} + \frac{1}{r} \frac{\partial W_0}{\partial r} - \frac{W_0}{r^2} \right) = \frac{\partial \Theta_0}{\partial z} \frac{\partial W_0}{\partial z} + \alpha^2 \frac{\partial \Theta_0}{\partial r} \left( \frac{\partial W_0}{\partial r} - \frac{W_0}{r} \right)$$

(15)

$$\frac{\partial^2 \Theta_0}{\partial z^2} + \alpha^2 \left( \frac{\partial^2 \Theta_0}{\partial r^2} + \frac{1}{r} \frac{\partial \Theta_0}{\partial r} \right) \theta = 0$$

(16)

with the boundary conditions,

- at $z = 0$, $W_0 = 0$, $\Theta_0 = \vartheta_w$
- at $z = 1$, $W_0 = r$, $\frac{\partial \Theta_0}{\partial z} + B \Theta_0 = B \vartheta_w$
- at $r = 0$, $W_0 = 0$, $|\Theta_0| < \infty$
- at $r = 1$, $\frac{\partial W_0}{\partial r} - \frac{W_0}{r} = 0$, $\frac{\partial \Theta_0}{\partial r} = 0$

The leading order solution for $Na = 0$ gives the isothermal solution

$$W_0 = rz,$$

(21)

$$\Theta_0 = \vartheta_w$$

(22)

Note that this solution is valid for all values of $\alpha$.

The solution at order $Na$ corresponding the first non-isothermal correction satisfies the following equations.

$$\frac{\partial^2 W_1}{\partial z^2} + \alpha^2 \left( \frac{\partial^2 W_1}{\partial r^2} + \frac{1}{r} \frac{\partial W_1}{\partial r} - \frac{W_1}{r^2} \right) = r \frac{\partial \Theta_1}{\partial z}$$

(23)

$$\frac{\partial^2 \Theta_1}{\partial z^2} + \alpha^2 \left( \frac{\partial^2 \Theta_1}{\partial r^2} + \frac{1}{r} \frac{\partial \Theta_1}{\partial r} \right) = -r^2$$

(24)

with boundary conditions,

- at $z = 0$, $W_1 = 0$, $\Theta_1 = 0$
- at $z = 1$, $W_1 = 0$, $\frac{\partial \Theta_1}{\partial z} + B \Theta_1 = 0$

(26)
at \( r = 0, \quad W_1 = 0, \quad |\Theta_1| < \infty \) \quad (27)

at \( r = 1, \quad \frac{\partial W_1}{\partial r} - \frac{W_1}{r} = 0, \quad \frac{\partial \Theta_1}{\partial r} = 0 \) \quad (28)

Note that equations (23) - (24) are now uncoupled. The equations can be solved exactly by separation of variables as follows.

\[
\Theta_1 = -\sum_{n=1}^{\infty} \frac{\bar{\Gamma}_n}{\lambda_n^2} \left[ 2\alpha^3 \frac{I_0(\lambda_n z)}{\lambda_n} - 4\alpha^4 \frac{\lambda_n}{\lambda_n^2} - \alpha^2 r^2 \right] \sin(\lambda_n z) \quad (29)
\]

and

\[
W_1 = \sum_{m=1}^{\infty} F_m(r) \sin(m\pi z). \quad (30)
\]

where \( \lambda_n, n = 1, 2, \cdots \) are positive solutions of the transcendental equation

\[
\tan(\lambda_n) + \frac{\lambda_n}{B} = 0, \quad (31)
\]

and \( \bar{\Gamma}_n = -\frac{4(1-\cos(\lambda_n))}{\pi^2 2m^2 \sin(2\lambda_n)}. \) This equation has infinitely many positive roots. Here \( I_n \) is the modified Bessel function of the first kind. This solution is also valid for all values of the aspect ratio \( \alpha. \)

The expression for \( F_m(r) \) involves complicated integrals of Bessel functions (see the Appendix for details).

### 3.3 Asymptotic solution for \( Na \ll 1 : \quad \vartheta_w \neq \vartheta_a \)

The governing equations for \( W_0 \) and \( \Theta_0, W_1 \) and \( \Theta_1 \) are the same as in the previous section.

At zeroth order in \( Na \) we have the isothermal solution

\[
\Theta_0 = \chi z + \vartheta_w \text{ where, } \chi \equiv \frac{B(\vartheta_a - \vartheta_w)}{1 + B} \quad (32)
\]

\[
W_0 = \left( \frac{e^{\chi z} - 1}{e^\chi - 1} \right) r \quad (33)
\]

The equations at order \( Na \) are

\[
\frac{\partial^2 W_1}{\partial z^2} + \alpha^2 \left( \frac{\partial^2 W_1}{\partial r^2} + \frac{1}{r} \frac{\partial W_1}{\partial r} - \frac{W_1}{r^2} \right) = \chi r \frac{e^{\chi z}}{e^\chi - 1} \frac{\partial \Theta_1}{\partial z} + \chi \frac{\partial W_1}{\partial z} \quad (34)
\]

\[
\frac{\partial^2 \Theta_1}{\partial z^2} + \alpha^2 \left( \frac{\partial^2 \Theta_1}{\partial r^2} + \frac{1}{r} \frac{\partial \Theta_1}{\partial r} \right) = -\frac{\chi^2 r^2 e^{\chi z}}{(e^\chi - 1)^2} \quad (35)
\]

with boundary conditions,

\[
\text{on } z = 0, \quad W_1 = 0, \quad \Theta_1 = 0 \quad (36)
\]

\[
\text{on } z = 1, \quad W_1 = 0, \quad \frac{\partial \Theta_1}{\partial z} + B \Theta_1 = 0 \quad (37)
\]

\[
\text{on } r = 0, \quad W_1 = 0, \quad |\Theta_1| < \infty \quad (38)
\]

\[
\text{on } r = 1, \quad \frac{\partial W_1}{\partial r} - \frac{W_1}{r} = 0, \quad \frac{\partial \Theta_1}{\partial r} = 0 \quad (39)
\]
Although these equations can also be solved exactly the solutions are rather too complicated. Therefore, we will obtain the solution as an expansion in $\alpha$. This limit has applications in rheometric devices where $\alpha$ is typically less than 0.1. For the case $\alpha = O(1)$ the solution will be computed numerically. Because the limit $\alpha \to 0$ is singular we use the method of matched asymptotic expansion. Thus we seek an outer solution, an inner solution and then obtain a composite expansion for the solution by matching.

**Outer Solution**

For the outer solution we expand as follows.

$$W_1 = W_{10}^o + \alpha W_{11}^o + O(\alpha^2), \quad \Theta_1 = \Theta_{10}^o + \alpha \Theta_{11}^o + O(\alpha^2) \quad (40)$$

where the superscript $(o)$ refers to the outer solution.

The governing equations at zeroth order in $\alpha$ are,

$$\frac{\partial^2 W_{10}^o}{\partial z^2} = \chi r e^{\chi z} \frac{\partial \Theta_{10}^o}{\partial z} + \chi \frac{\partial W_{10}^o}{\partial z} \quad (41)$$

$$\frac{\partial^2 \Theta_{10}^o}{\partial z^2} = -\frac{\chi^2 r^2 e^{\chi z}}{(e^\chi - 1)^2} \quad (42)$$

with the corresponding boundary conditions,

$$at \ z = 0, \quad W_{10}^o = 0, \quad \Theta_{10}^o = 0 \quad (43)$$

$$at \ z = 1, \quad W_{10}^o = 0, \quad \frac{\partial \Theta_{10}^o}{\partial z} + B \Theta_{10}^o = 0 \quad (44)$$

The solution satisfying the above governing equations and the boundary conditions are,

$$\Theta_{10}^o = \frac{r^2}{(e^\chi - 1)^2} \left[1 - e^{\chi z} + z \frac{(\chi + B)e^\chi - B}{1 + B}\right] \quad (45)$$

$$W_{10}^o = \frac{\chi r^3}{(e^\chi - 1)^3} \left[-\frac{e^{2\chi z}}{2\chi} + \frac{(\chi + B)e^\chi - B}{1 + B} \frac{e^{\chi z}(\chi z - 1)}{(1 + B)^2}\right]$$

$$- \frac{\chi e^{\chi z} r^3}{(e^\chi - 1)^4} \left[1 - \frac{e^{2\chi}}{2\chi} + \frac{(\chi + B)e^\chi - B}{1 + B} \frac{1 + (\chi - 1)e^\chi}{(1 + B)^2}\right]$$

$$- \frac{r^3}{(e^\chi - 1)^4} \left[\frac{e^{2\chi} - e^\chi}{2} - \frac{e^\chi ((\chi + B)e^\chi - B)}{1 + B}\right] \quad (46)$$

Further, it is also determined that,

$$W_{11}^o = 0 \quad \Theta_{11}^o = 0. \quad (48)$$

**Inner solution**

For the inner expansion we introduce the stretched variable, $\xi = \frac{1-r}{\alpha}$, and seek an expansion of the form

$$W_1^i = W_{10}^i + \alpha W_{11}^i + O(\alpha^2), \quad \Theta_1^i = \Theta_{10}^i + \alpha \Theta_{11}^i + O(\alpha^2). \quad (49)$$
The governing equations and the boundary conditions at zeroth order in $\alpha$ are,

\[
\frac{\partial^2 W_{10}^i}{\partial z^2} + \frac{\partial^2 W_{10}^i}{\partial \xi^2} - \chi \frac{\partial W_{10}^i}{\partial z} = \frac{\chi e^{\chi z}}{(e^\chi - 1)} \left( \frac{\partial \Theta_{10}^i}{\partial z} \right)
\]

(50)

\[
\frac{\partial^2 \Theta_{10}^i}{\partial z^2} + \frac{\partial^2 \Theta_{10}^i}{\partial \xi^2} = -\frac{\chi^2 e^{\chi z}}{(e^\chi - 1)^2}
\]

(51)

at $z = 0$, $W_{10}^i = 0$, $\Theta_{10}^i = 0$

(52)

at $z = 1$, $W_{10}^i = 0$, $\frac{\partial \Theta_{10}^i}{\partial z} + B \Theta_{10}^i = 0$

(53)

at $\xi = 0$, $\frac{\partial W_{10}^i}{\partial \xi} = 0$, $\frac{\partial \Theta_{10}^i}{\partial \xi} = 0$

(54)

It is straightforward to obtain the following expressions,

\[
\Theta_{10}^i = \frac{1}{(e^\chi - 1)^2} \left[ 1 - e^{\chi z} + \frac{z (\chi + B)e^\chi - B}{1 + B} \right]
\]

(55)

\[
W_{10}^i = \frac{\chi}{(e^\chi - 1)^3} \left[ -\frac{e^{2\chi z}}{2\chi} + \frac{(\chi + B)e^\chi - B}{e^\chi - 1} \frac{\chi^2 (\chi z - 1)}{(1 + B)^2} \right] - \frac{\chi e^{\chi z}}{(e^\chi - 1)^4} \left[ -\frac{1 - e^{2\chi z}}{2\chi} + \frac{(\chi + B)e^\chi - B}{1 + (\chi - 1)e^\chi} \frac{1 + (\chi + B)e^\chi}{(1 + B)^2} \right] - \frac{1}{(e^\chi - 1)^4} \left[ -\frac{e^{2\chi z} - e^\chi}{2} - \frac{\chi^2 - (\chi + B)e^\chi - B}{1 + B} \right]
\]

(56)

(57)

The order $\alpha$ equations are

\[
\frac{\partial^2 \Theta_{11}^i}{\partial z^2} + \frac{\partial^2 \Theta_{11}^i}{\partial \xi^2} = \frac{2\xi \chi^2 e^{\chi z}}{(e^\chi - 1)^2}
\]

(58)

\[
\frac{\partial^2 W_{11}^i}{\partial z^2} + \frac{\partial^2 W_{11}^i}{\partial \xi^2} - \chi \frac{\partial W_{11}^i}{\partial z} = \frac{\partial W_{10}^i}{\partial z} \left( \frac{\partial \Theta_{10}^i}{\partial z} \right) + \frac{\partial W_{10}^i}{\partial z} \left( \frac{\partial \Theta_{10}^i}{\partial z} \right)
\]

(59)

The boundary conditions are the same as above. The solution of the equations is

\[
\Theta_{11}^i = -2\xi \Theta_{10}^i - 2 \sum_{n=1}^{\infty} \tilde{\Gamma}_n e^{-\Lambda_n (1-\xi)} \frac{\Lambda_n}{\lambda_n^3} \sin(\lambda_n z),
\]

(60)

\[
W_{11}^i = -3\xi W_{10}^i + \sum_{m=1}^{\infty} \frac{2 \tilde{A}_m e^{-\Lambda_m \xi} e^{\chi z}}{\Lambda_m^2} \sin(m\pi z)
\]

(61)

\[+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\tilde{\Gamma}_n B_{mn} \Lambda_m}{\lambda_n^4 (\Lambda_m^2 - \lambda_n^2)} \left( e^{-\lambda_n \xi} - \frac{\lambda_n e^{-\Lambda_m \xi}}{\Lambda_m} \right) e^{\chi z} \sin(m\pi z),\]

where,
\[
\hat{\Gamma}_n = \frac{\chi^2 \int_0^1 e^{\chi z} \sin(\lambda_n z) dz}{(e^\chi - 1)^2 \int_0^1 \sin^2(\lambda_n z) dz}
\]

and,

\[
\begin{align*}
\lambda_m^2 &= \frac{\chi^2}{4} + m^2 \pi^2 \\
\tilde{A}_m &= \frac{\chi}{(e^\chi - 1)^3} \frac{\int_0^1 (e^{\chi z} + \frac{(\chi + B)e^\chi - B}{1 + B}) e^{\chi z} \sin(m\pi z) dz}{\int_0^1 \sin^2(m\pi z) dz} \\
\tilde{B}_{mn} &= \frac{2\chi}{e^\chi - 1} \frac{\int_0^1 e^{\chi z} \cos(\lambda_n z) \sin(m\pi z) dz}{\int_0^1 \sin^2(m\pi z) dz}
\end{align*}
\]

**Composite solution**

In order to use the Van Dyke’s matching principle, the outer solution for the velocity and the temperature distribution is expressed in terms of the inner variable including terms of the first order in \(\alpha\),

\[
(\Theta^o)^i = \chi z + \theta_w + Na(1 - 2\alpha \xi)\Theta^1_{i0} + O(Na^2)
\]

\[
(W^o)^i = \left(\frac{e^{\chi z} - 1}{e^\chi - 1}\right) r + Na(1 - 3\alpha \xi)W^1_{i0} + O(Na^2)
\]

The Van Dyke’s matching principle is expressed using the formulas,

\[
W^c = W^o + W^i - (W^o)^i, \quad \Theta^c = \Theta^o + \Theta^i - (\Theta^o)^i
\]

The composite solution for temperature and velocity distribution is then given by,

\[
\Theta^c = \chi z + \theta_w + Na \left(\frac{r^2}{(e^\chi - 1)^2} \left[1 - e^{\chi z} + \frac{(\chi + B)e^\chi - B}{1 + B}\right]\right)
-2Na \alpha \sum_{n=1}^{\infty} \frac{\hat{\Gamma}_n e^{-\lambda_n (1 - r) \alpha}}{\lambda_n^3} \sin(\lambda_n z) + O(\text{Na}^2)
\]

\[
W^c = \left(\frac{e^{\chi z} - 1}{e^\chi - 1}\right) r + Na W^1_{i1} + Na \alpha \sum_{m=1}^{\infty} \frac{2\tilde{A}_m e^{-\Lambda_m (1 - r) \alpha}}{\Lambda_m^2} e^{\chi z} \sin(m\pi z)
+ Na \alpha \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\hat{\Gamma}_n \tilde{B}_{mn}}{\lambda_n^2 (\Lambda_m^2 - \lambda_n^2)} \left(e^{-\lambda_n (1 - r) \alpha} - \frac{\lambda_n e^{-\Lambda_m (1 - r) \alpha}}{\Lambda_m}\right) e^{\chi z} \sin m\pi z
\]
4 Numerical solution

The domain \( \Omega \) for numerical computation is \( 0 < z < 1 \) and \( 0 < r < 1 \), shown in Figure 1. In order to apply the finite element method, we need to rewrite the two PDEs in variational forms. We multiply the continuity equation (1) by \( rV(r,z) \), cf. [22, 24, 25, 26], the test function with boundary conditions specified in Figure 1. Then we apply the integration by parts to obtain

\[
\int_{\Omega} \left( \frac{\partial W}{\partial z} \frac{\partial V}{\partial z} + \alpha^2 \frac{\partial W}{\partial r} \frac{\partial V}{\partial r} + \frac{\alpha^2}{r^2} W V \right) r dr dz - \int_{r=1} \alpha^2 W V dz = \int_{\Omega} \left( - \frac{\partial \Theta}{\partial z} \frac{\partial W}{\partial z} - \alpha^2 \frac{\partial \Theta}{\partial r} \left( \frac{\partial W}{\partial r} - \frac{W}{r} \right) \right) V r dr dz. \tag{69}
\]

We do the same for the second equation (2), with a test function \( rV(r,z) \), but of different boundary conditions shown in Figure 1.

\[
\int_{\Omega} \left( \frac{\partial \Theta}{\partial z} \frac{\partial V}{\partial z} + \alpha^2 \frac{\partial \Theta}{\partial r} \frac{\partial V}{\partial r} \right) r dr dz - \int_{z=1} r(B\vartheta_a - B\Theta)V dr = Na \int_{\Omega} e^{-\Theta} \left( \frac{\partial W}{\partial z} \right)^2 + \alpha^2 \left( \frac{\partial W}{\partial r} - \frac{W}{r} \right)^2 V r dr dz. \tag{70}
\]

To obtain homogeneous boundary conditions for the variational problems (69)–(70), we use the following decompositions

\[
W = W^b + W^0, \quad W^b = rz, \tag{71}
\]

\[
\Theta = \Theta^b + \Theta^0, \quad \Theta^b = \vartheta_w + z \frac{B}{1 + B} (\vartheta_a - \vartheta_w). \tag{72}
\]

We seek solutions \( W^0 \) and \( \Theta^0 \) instead, which have homogeneous boundary conditions, also depicted in Figure 2,

\[
W^0 \bigg|_{r=0, z=0, z=1} = 0, \quad \left. \frac{\partial W^0}{\partial r} \right|_{r=1} = \frac{W^0}{r},
\]

\[
\Theta^0 \bigg|_{z=0} = 0, \quad \left. \frac{\partial \Theta^0}{\partial r} \right|_{r=0, r=1} = 0, \quad \left. \frac{\partial \Theta^0}{\partial z} \right|_{z=1} = -B\Theta^0.
\]

That is, we will find finite element solutions \( W^0_h \) and \( \Theta^0_h \) where \( h \) stands for the grid size.

To discretize (69) and (70), due to the special domain of the unit square, one may use spectral methods (cf. [22]) or tensor product methods (cf. [23]) to get a high order approximation.
To handle the nonlinearity of the coupled system, and to handle possible irregular domains in future, we use $Q^k$ finite elements, continuous and piecewise polynomials of separate degree $k$ or less, on uniform grids $K_h = \{ K \mid K = [r_i - h, r_i] \times [z_j - h, z_j], \ i,j = 1, \ldots, 1/h \}$ of $\Omega$:

$$Q_h \ := \ \left\{ V \in C(\Omega) \mid V|_K = \sum_{0 \leq i,j \leq k} a_{ij} r^i z^j, \forall K \in K_h \right\} \subset H^1(\Omega).$$

We use the following notations for the discrete spaces with homogeneous boundary conditions:

$$Q_{h,W} := Q_h \cap \{ V = V(r,z) \in C(\Omega) \mid V(0,z) = V(r,0) = V(r,1) = 0 \},$$

$$Q_{h,\Theta} := Q_h \cap \{ V = V(r,z) \in C(\Omega) \mid V(r,0) = 0 \}.$$  

(73)  

(74)

The finite element discretizations of (69)–(70) read: Find $(W_h^0, \Theta_h^0) \in Q_{h,W} \times Q_{h,\Theta}$ such that

$$A_W(W_h^0, V) = F_{W,\Theta}(V) - A_W(W_h^b, V), \quad \forall V \in Q_{h,W},$$

(75)

$$A_\Theta(\Theta_h^0, V) = G_{W,\Theta}(V) - A_\Theta(\Theta_h^b, V) + Bc_\theta(\vartheta_a, V), \quad \forall V \in Q_{h,\Theta}.$$  

(76)
where the bilinear forms and functionals are defined by

\[ A_W(U, V) = a(U, V) + \alpha^2 \left( \frac{U}{r}, \frac{V}{r} \right)_r - \alpha^2 c_r(U, V), \]

\[ A_{\Theta}(U, V) = a(U, V) + Bc_z(U, V), \]

\[ F_{W, \Theta}(V) = \left( -\frac{\partial \Theta}{\partial z} \frac{\partial W}{\partial z} - \alpha^2 \frac{\partial \Theta}{\partial r} \left( \frac{\partial W}{\partial r} - \frac{W}{r} \right)_r, V \right)_r, \]

\[ G_{W, \Theta}(V) = Na \left( e^{-\Theta} \left( \frac{\partial W}{\partial z} \right)^2 + \alpha^2 \left( \frac{\partial W}{\partial r} - \frac{W}{r} \right)_r^2 \right)_r. \]

We solve the nonlinear system of equations (75)–(76) numerically by a straightforward Seidel iteration. That is, given initially some guesses (both zero in computation) of \( W_h^0 \) and \( \Theta_h^0 \), we generate the right hand side of (75) and use the conjugate gradient method to solve (75) to get a new \( W_{h,j}^0 \). Then the new \( W_{h,j}^0 \) and the old \( \Theta_h^0 \) would be used to generate the right hand side vector in (76). We solve (76) again by the conjugate gradient method to get a new \( \Theta_h^0 \).

The next lemma shows that the two linear systems at each step described above are uniquely solvable, because both the coefficient matrices are symmetric and positive definite.

**Lemma 4.1** For any \( V \in Q_{h,T} \cup Q_{h,W} \) and \( V \neq 0 \),

\[ a(V, V) > 0. \]

For any \( V \in Q_{h,W} \) and \( V \neq 0 \),

\[ A_W(V, V) > 0. \]

For any \( V \in Q_{h,T} \), \( V \neq 0 \), and \( B \geq 0 \),

\[ A_{\Theta}(V, V) > 0. \]

**Proof.** (85) and (86) are shown in [27]. (87) is a corollary of (85), noting the sign of \( B \) is positive.

**Algorithm 4.1** The coupled nonlinear system (75)–(76) is solved by the Seidel iteration with the given initial guess \( W_{h,0}^0 = 0 \) and \( W_{\Theta,0}^0 = 0 \). For \( j = 1, 2, \ldots \),

\[ W_{h,j}^0 = W_{h,j-1}^0 + e_W, \]

where \( e_W \) solves the equation

\[ A_W(e_W, V) = F_{W_{j-1}, \Theta_{j-1}}(V) - A_W(W_h^j, V) - A_W(W_{h,j-1}^0, V) \quad \forall V \in Q_{h,W}, \]

and

\[ c_r(U, V) = \int_{r=1, \theta \leq z \leq 1} UV dz, \]

\[ c_z(U, V) = \int_{z=1, \theta \leq r \leq 1} rUV dz. \]
\[ \Theta_{h,j}^0 = \Theta_{h,j-1}^0 + \epsilon_\Theta, \]

where \( \epsilon_\Theta \) solves the equation

\[ A_\Theta(\epsilon_\Theta, V) = G_{W_j, \Theta_{j-1}}(V) - A_\Theta(\Theta^b, V) + B_{\Theta - g_a}(V) - A_\Theta(\Theta^0_{h,j-1}, V) \quad \forall V \in Q_{h,\Theta}. \quad (89) \]

Here \( W_j = W^b + W^0_{h,j} \) and \( \Theta_j = \Theta^b + \Theta^0_{h,j} \) for \( j = 0, 1, 2, \ldots \).

A typical pair of solutions \((W_h, \Theta_h)\) is shown in Figure 3.

Figure 3: Solutions \(W\) and \(\Theta\) for (1) and (2) when \(\alpha = .01\), \(Na = 1\), \(\vartheta_w = 1.5\), \(\vartheta_a = 1\) \(B = 0.1\).

Figure 4: Solutions obtained by (11)–(12), (29)–(30), and (75)–(76).
Figure 5: Solutions obtained by (11)–(12), (29)–(30), and (75)–(76).

Figure 6: Solutions obtained by (11)–(12), (29)–(30), and (75)–(76).
At \( r = 0.5 \) for \( \alpha = 0.1, \theta_a = 1, \theta_w = 1.5, B = 1 \).

(a) Numerical solution \( Na = 0.1 \)
(b) Asymptotic solution \( Na = 0.1 \)
(c) Numerical solution \( Na = 1 \)
(b) Asymptotic solution \( Na = 1 \)

Figure 7: Solutions obtained by (11)–(12), (29)–(30), and (75)–(76).
5 Discussion

In this section we compare the analytical solutions obtained in section 3 with the finite element solution obtained in section 4. The exact solution given in section 3.1 equations (11)-(12) is valid for $\alpha = 0$, $B = 0$ and all values of $Na$, the perturbation solution given in section 3.2 equations (29)-(30) is valid for all values of $\alpha$ and small $Na$ while the solution found in section 3.3 equations (67)-(68) is valid only for small $\alpha$ and $Na$. The numerical solution on the other hand is valid for all parameter values.

![Figure 8: Solutions obtained by (11)–(12), (29)–(30), and (75)–(76).](image)

The plots in Figs 4-8 depict the deviation of the temperature and velocity from the isothermal solution. We plot $\Theta - \Theta_0$ and $W - W_0$ where $\Theta_0$ and $W_0$ are the isothermal solutions. Figs. 4 and 5 show the deviation of temperature at $z = 0.5$ and $r = 0.5$ respectively for $Na = 0.1$, $B = 0$ and selected values of $\alpha$. The case $B = 0$ corresponds to insulated boundary condition on the upper plate. For $\alpha = 0.1$ all three solutions agree very well except near the free surface $r = 1$. The error in the exact solution for $\alpha = 0$ arises because it does not satisfy the boundary condition at the free surface. The error between the numerical and asymptotic solutions is otherwise very small. Figure 6 shows the deviation of the velocity for same values of the parameters. The agreement among all three solutions is again very good. In Figure 7, the deviation in temperature is shown for $\alpha = 0.1$, $B = 1$, $\theta_w = 1.0$, $\theta_a = 1.5$ for two values of $Na$. While the agreement between numerical and asymptotic solutions is excellent for $Na = 0.1$, there is a small discrepancy for the case $Na = 1.0$. This is actually quite good since the perturbation expansion was truncated at order $O(\text{Na})$.

Figure 8 shows the deviation of the velocity for a large value of the Biot number $B$. In this limit the two plates are nearly isothermal and we see a qualitatively different from the solution for $B = O(1)$. Specifically, the profile is symmetric about the mid-plane $z = 0.5$.
whereas for smaller values of $B$ the profile is asymmetric. Another qualitative difference is the location of the maximum temperature. On any fixed plane the maximum occurs at the free surface $r = 1$ when $B = 0$. As $B$ increases the location of the maximum moves away from the free surface. Form these figures we also see that the deviation in of temperature velocity from isothermal is order $O(Na)$ when $Na$ is small.

Lastly, in Figure 9 we plot the torque on the lower stationary plate as a function of the Nahme number $Na$ for selected values of $\alpha$ and $B$. Although the exact solution is valid only for $\alpha = 0$ and $B = 0$ the agreement with the numerical solution for $\alpha = 0.1$ and $B = 0.1$ is excellent. Thus, in applications in which the aspect ratio $\alpha$ is small the exact solution can be used to obtain very accurate corrections to the torque in viscometric applications. We also show the series representation for the torque equation (13) and the agreement is excellent for $Na < 2$.

6 Summary

Non-isothermal torsional flow with heat transfer boundary condition at the upper rotating plate, isothermal boundary condition at the lower stationary plate, and insulated boundary condition at the fluid/air interface has been analyzed. It is assumed that viscosity in an exponential function of temperature. We have obtained analytical solutions valid in the limit of small aspect ratio $\alpha$ and in the limit of small Nahme-Griffith number $Na$. The nonlinear coupled partial differential equations have also been solved numerically using the finite element method. Our results show that the asymptotic solutions agree very well with the numerical solution. For small values of $Na$ the deviation of temperature and velocity from the isothermal solution is small approximately of order $O(Na)$. Furthermore, we show that for viscometric applications in which the aspect ratio $\alpha$ is typically less than 0.1, the exact solution obtained
for $\alpha = 0$ and $B = 0$ gives very accurate results for the non-isothermal correction to the torque for small values of $\alpha$ and the Biot number $B$.

A Appendix

A.1 Heat transfer boundary condition

In this section, the derivation of the heat transfer boundary condition at the upper rotating plate is detailed. Following the convention adopted in Ozisik [29],

$$-k \frac{\partial \tilde{T}}{\partial z} = h \left( \tilde{T} - \tilde{T}_a \right), \quad \text{at the surface } S \quad (90)$$

where, $h$ is the heat transfer coefficient and $k$ is the thermal conductivity of the rotating plate. The surface $S$ corresponds to the surface of the upper rotating plate at $z = 1$. After normalization of variables and introducing the thickness of the plate $H$, to procure a meaningful parameter, the Biot number $Bi = \frac{hH}{k}$.

$$\frac{\partial T}{\partial z} = \frac{Bi h}{H} (-T + T_a) \quad (91)$$

where, $h$ is the thickness of the gap between the two plates [28, 29]. In dimensionless form this becomes

$$\frac{\partial \Theta}{\partial z} + B \Theta = B \vartheta_a \quad (92)$$

where, $B \equiv \frac{Bi h}{H}$.

A.2 The function $F_m(r)$

The function $F_m(r)$ appearing in equation (30) satisfies the following ordinary differential equation

$$r^2 F_m'' + r F_m' - \left(1 + \frac{m^2 \pi^2 r^2}{\alpha^2} \right) F_m = \varphi(r)$$

$$\varphi = 2r^3 \sum_{n=1}^{\infty} \left[ \frac{m \pi \Gamma_n \left(1 - \cos(m \pi) \cos(\lambda_n)\right)}{m^2 \pi^2 - \lambda_n^2} \right] \left[ \frac{2\alpha I_0 \left(\frac{\lambda_n r}{\alpha}\right)}{\lambda_n^2 I_1 \left(\frac{\lambda_n}{\alpha}\right)} - \frac{r^2}{\lambda_n^2} - \frac{4\alpha^2}{\lambda_n^4} \right] \quad (93)$$

The general solution to the above ordinary differential equation is,

$$F_m(r) = C_1 I_1 \left(\frac{m \pi r}{\alpha}\right) + C_2 K_1 \left(\frac{m \pi r}{\alpha}\right) + I_1 \left(\frac{m \pi r}{\alpha}\right) \left[ \int \frac{\varphi(r) K_1 \left(\frac{m \pi r}{\alpha}\right)}{r} \, dr \right]$$

$$-K_1 \left(\frac{m \pi r}{\alpha}\right) \left[ \int \frac{\varphi(r) I_1 \left(\frac{m \pi r}{\alpha}\right)}{r} \, dr \right], \quad (94)$$

where $K_0$ and $K_1$ are modified Bessel functions of the second kind.

The evaluation of the integrals are shown next.
\[
\int \frac{\varphi(r)K_1 \left( \frac{m\pi r}{\alpha} \right)}{r} \, dr = 2 \sum_{n=1}^{\infty} \frac{m\pi \Gamma_n (1 - \cos(m\pi) \cos(\lambda_n))}{\lambda_n(m^2 \pi^2 - \lambda_n^2)} \left[ -\frac{4m\pi r\alpha^3}{\lambda_n(m^2 \pi^2 - \lambda_n^2)^2} \right. \\
\left. \times \frac{\lambda_n I_1 \left( \frac{m\pi r}{\alpha} \right) K_0 \left( \frac{m\pi r}{\alpha} \right) + m\pi I_0 \left( \frac{m\pi r}{\alpha} \right) K_1 \left( \frac{m\pi r}{\alpha} \right)}{I_1 \left( \frac{m\pi r}{\alpha} \right)} - \frac{2r^2\alpha^2}{\lambda_n(m^2 \pi^2 - \lambda_n^2)} \right. \\
\left. \times \frac{\lambda_n I_1 \left( \frac{m\pi r}{\alpha} \right) K_1 \left( \frac{m\pi r}{\alpha} \right) + m\pi I_0 \left( \frac{m\pi r}{\alpha} \right) K_0 \left( \frac{m\pi r}{\alpha} \right)}{I_1 \left( \frac{m\pi r}{\alpha} \right)} + \frac{\alpha^4}{m\pi} K_2 \left( \frac{m\pi r}{\alpha} \right) \right] + \frac{2\alpha^2 r^3}{m^2 \pi^2} K_3 \left( \frac{m\pi r}{\alpha} \right) + \frac{4\alpha^3 r^2}{m\pi \lambda_n^2} K_2 \left( \frac{m\pi r}{\alpha} \right) \right] 
\]

(95)

\[
\int \frac{\varphi(r)I_1 \left( \frac{m\pi r}{\alpha} \right)}{r} \, dr = 2 \sum_{n=1}^{\infty} \frac{m\pi \Gamma_n (1 - \cos(m\pi) \cos(\lambda_n))}{\lambda_n(m^2 \pi^2 - \lambda_n^2)} \left[ \frac{4m\pi r\alpha^3}{\lambda_n(m^2 \pi^2 - \lambda_n^2)^2} \right. \\
\left. \times \frac{\lambda_n I_0 \left( \frac{m\pi r}{\alpha} \right) I_1 \left( \frac{m\pi r}{\alpha} \right) - m\pi I_0 \left( \frac{m\pi r}{\alpha} \right) I_1 \left( \frac{m\pi r}{\alpha} \right)}{I_1 \left( \frac{m\pi r}{\alpha} \right)} - \frac{2r^2\alpha^2}{\lambda_n(m^2 \pi^2 - \lambda_n^2)} \right. \\
\left. \times \frac{\lambda_n I_1 \left( \frac{m\pi r}{\alpha} \right) I_1 \left( \frac{m\pi r}{\alpha} \right) - m\pi I_0 \left( \frac{m\pi r}{\alpha} \right) I_0 \left( \frac{m\pi r}{\alpha} \right)}{I_1 \left( \frac{m\pi r}{\alpha} \right)} + \frac{\alpha^4}{m\pi} I_2 \left( \frac{m\pi r}{\alpha} \right) \right] + \frac{2\alpha^2 r^3}{m^2 \pi^2} I_3 \left( \frac{m\pi r}{\alpha} \right) - \frac{4\alpha^3 r^2}{m\pi \lambda_n^2} I_2 \left( \frac{m\pi r}{\alpha} \right) \right] 
\]

(96)

The boundary conditions, \( F_m = 0 \) at \( r = 0 \), and \( F'_m - \frac{F_m}{r} = 0 \) at \( r = 1 \) are used to determine the constants in the above equation. The boundary condition on the axis causes the constant of integration \( C_2 \) to be zero. The other boundary condition is used to determine the constant of integration \( C_1 \). However, because of the complexity of the nature of the solution as judged from the above equations, the constant \( C_1 \) is shown determined in principle. However, the actual formulation is employed to generate plots via CAS (Computer Algebra System).

\[
C_1 = - \int_0^1 \frac{\varphi(r)K_1 \left( \frac{m\pi r}{\alpha} \right)}{r} \, dr - \frac{K_0 \left( \frac{m\pi r}{\alpha} \right) + K_2 \left( \frac{m\pi r}{\alpha} \right) - 2\alpha \frac{m\pi}{r} K_1 \left( \frac{m\pi r}{\alpha} \right)}{I_0 \left( \frac{m\pi r}{\alpha} \right) + I_2 \left( \frac{m\pi r}{\alpha} \right) - 2\alpha \frac{m\pi}{r} I_1 \left( \frac{m\pi r}{\alpha} \right)} \left[ \int_0^1 \frac{\varphi(r)I_1 \left( \frac{m\pi r}{\alpha} \right)}{r} \, dr \right] \right|_{r=1} 
\]

(97)

References


