On the Multigrid Method for 3D Mixed Macro-Elements

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Abstract

A natural mixed finite element method for the Stokes problem in the velocity-pressure formulation is to approximate the velocity by continuous piecewise polynomials of degree \((k + 1)\) and to approximate the pressure by discontinuous piecewise polynomials of degree \(k\). This paper is devoted to prove constant rates of convergence for two nonnested multigrid methods when applied to solve some 3D stable \(P_{k+1}-P_k\) mixed-element equations where the underlying tetrahedral meshes have a macro-element structure. A numerical test is presented.

Key words: Multigrid method, Stokes equations, mixed element, macro-element, tetrahedral mesh.


1 Introduction

In the variation form of velocity-pressure formulation of the Stokes equations, the velocity and pressure are in the Sobolev spaces \(H^1(\Omega)^d\) and \(L^2_0(\Omega)\), respectively. The mixed element approximation spaces can be chosen to be the corresponding subspaces. A most natural approximation scheme would be then to choose continuous piecewise-polynomials of degree \((k + 1)\) for the velocity and discontinuous piecewise-polynomials of degree \(k\) for the pressure. Such mixed element solutions satisfy the incompressibility condition. Scott and Vogelius [8] showed that the Babuška-Brezzi inequality holds for such \(P_{k+1}-P_k\) triangular mixed-elements in 2D if the polynomial degree \(k\) is 3 or a higher and if the meshes are singular-vertex free. This result is partially extended to 3D in [15]. It is shown that, when defined on tetrahedral meshes of a macro-element type, the above \(P_{k+1}-P_k\) elements are stable if the polynomial degree for velocity is 3 or higher. In the method, starting with any quasi-uniform tetrahedral mesh, the mesh for computation is generated by subdividing each initial tetrahedron into 4 subtetrahedra by connecting the bary-center with 4 vertices (see Figure 1). The \(P_{k+1}-P_k\) mixed elements are defined on this new mesh. The velocity in the mixed-element solution is divergence free pointwise.

![Figure 1: A macro-element consists of 4 tetrahedra.](image)

The multigrid method is an effective iteration method for solving linear systems of equations arising from discretizing partial differential equations. It is an optimal order algorithm in various cases ([1], [2] and references therein). Verfürth has introduced two multigrid methods ([12] and [13]) for solving mixed element equations for Stokes problems and proved that the iterations converge with constant rates (independent of the number of unknowns in the linear system of equations). But the standard (nested) multigrid algorithm will not work for 3D \(P_{k+1}-P_k\) tetrahedral mixed-elements. This is because the stability condition (see (2) below) will not hold any more at the multigrid refinement as the new interior edge of each tetrahedron is singular when the tetrahedron is subdivided into 8 half-sized tetrahedra (cf. [15]). Now, to get rid of singularity, if we cut each of the 8 subtetrahedra further into 4 as depicted in Figure 1, then this process would lead to degenerate meshes which contain sharp and long
tetrahedra. Given an initial tetrahedral grid, the correct way to build multilevel grids is to refine the initial grid nestedly to the highest level first. Then we can subdivide all tetrahedra on all levels to get the macro-element meshes. However, the multilevel grids created this way are not nested and the resulting mixed element spaces are not nested either. Some treatments are needed in defining the intergrid transfer operator which is necessary in the multigrid method to transfer functions from a lower level to higher one (cf. [3] and [17]). A multigrid method for 2D, biharmonic, $C^1$ elements is studies in [16], where the meshes have the same macro-element structure.

In this note, we apply the two multigrid methods of Verfürth to solve the new mixed-element equations and prove that the two methods retain their constant-rate of convergence and their optimal order of computation, consequently. Section 2 provides some basics on the mixed element approximation of the stationary Stokes equations. We define a multigrid method for the mixed-element equations in Section 3 and prove its constant rate of convergence in Section 4. In Section 5, a combined conjugate gradient – multigrid method is defined, which converges also with a constant rate. A numerical test is presented in Section 6.

2 Preliminaries

We consider the stationary Stokes problem: Find functions $\mathbf{u}$ (the fluid velocity) and $p$ (the pressure) on a 3D domain $\Omega$ such that

$$
-\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \\
\text{div} \, \mathbf{u} = 0 \quad \text{in } \Omega, \\
\mathbf{u} = 0 \quad \text{on } \partial \Omega,
$$

where $\mathbf{f}$ is the body force and $\nu$ is the kinematic viscosity.

Given an initial quasi-uniform (cf. [4]) tetrahedral mesh $\mathcal{M}_0 = \{ \mathcal{T} \}$ with mesh size $h_0$ on $\Omega$, we can refine each tetrahedron $\mathcal{T}$ into 8 subtetrahedra (cf. [18]) nestedly to $\mathcal{T}_j$ for $j = 0, 1, \cdots$. Let $\mathcal{T}_j = \{ \mathcal{T} \}$ denote the corresponding macro-element mesh where each tetrahedron $\mathcal{T}$ of $\mathcal{M}_j$ is cut into 4 by connecting the bary-center with 4 vertices as depicted in Figure 1. Let $\mathcal{P}_{k,\mathcal{T}_j}$ and $\mathcal{P}_{k,\mathcal{T}_j}^0$ denote the piecewise continuous and discontinuous polynomials of degree $k$ on the mesh $\mathcal{T}_j$, respectively. Let $\mathcal{P}_{k,\mathcal{T}_j}^0 = \mathcal{P}_{k,\mathcal{T}_j} \cap H_0^1(\Omega)$ and $\mathcal{P}_{k,\mathcal{T}_j}^0 = \mathcal{P}_{k,\mathcal{T}_j} \cap L_0^2(\Omega)$, i.e., $\mathcal{P}_{k,\mathcal{T}_j}^0 = \{ p \in \mathcal{P}_{k,\mathcal{T}_j} \mid \int_\Omega p = 0 \}$.

To shorten the notation, we let $V_j = (\mathcal{P}_{k,\mathcal{T}_j+1}^0)^3 \times \mathcal{P}_{k,\mathcal{T}_j}^0$ and we will mention the dependence on the polynomial degree $k$ when needed. It is shown in [15] that $V_j$ satisfies the Babuška-Brezzi stability condition: there exists a constant $C > 0$ (independent of $j$, but depending on $k$) such that

$$(2) \quad \sup_{\mathbf{v} \in (\mathcal{P}_{k+1,\mathcal{T}_j}^0)^3, \mathbf{v} \neq 0} \frac{\text{div} \, \mathbf{v} \cdot p}{|\mathbf{v}|} \geq C \| p \|_0 \quad \forall p \in \mathcal{P}_{k,\mathcal{T}_j}^0,$$

which ensures the best order of convergence for the mixed elements solutions $\{ \{ \mathbf{u}_j, p_j \} \}$ defined below in (3). In this paper, we use the standard notation for Sobolev spaces and their norms, and we use $C$ as a generic constant.

The mixed elements approximation to (1) in weak formulation is: Find $\{ \mathbf{u}_j, p_j \} \in V_j$, such that

$$
(3) \quad L([\mathbf{u}_j, p_j], [\mathbf{v}, q]) = (\mathbf{f}, \mathbf{v}) \quad \forall [\mathbf{v}, q] \in V_j,
$$

where $L([\mathbf{u}, p], [\mathbf{v}, q]) := a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b(\mathbf{q}, q)$, $a(\mathbf{u}, \mathbf{v}) := \nu(\nabla \mathbf{u} \cdot \nabla \mathbf{v})$ and $b(\mathbf{v}, p) := -\text{div}(\mathbf{v} p)$. We assume the boundary of $\Omega$ is regular enough such that if $\mathbf{f} \in L^2(\Omega)$ in (1), then the solution $[\mathbf{u}, p] \in H^2(\Omega)^3 \times H^1(\Omega)$ (cf. [5], [6] and [11]) and

$$
(4) \quad \| \mathbf{u} \|_2 + \| p \|_1 \leq C \| \mathbf{f} \|_0.
$$

The analysis here can be extended to cover some domains with curved boundary where (4) is known to hold. By (2) and (4), it follows that ([5], [4])

$$
(5) \quad \| \mathbf{u}_j - \mathbf{u} \|_1 + \| p_j - p \|_0 \leq C h_j \| \mathbf{f} \|_0, \\
\| \mathbf{u}_j - \mathbf{u} \|_0 \leq C h_j^2 \| \mathbf{f} \|_0.
$$

3 A multigrid algorithm

In this section we introduce a multigrid algorithm for solving the mixed-element equations (3). The algorithm is based on the general multigrid algorithm defined by Verfürth in [12]. We need to solve (3) on the highest level. Problems on all lower levels are auxiliary ones. We rewrite (3) in a more general form

$$
(6) \quad L([\mathbf{u}_j, p_j], [\mathbf{v}, q]) = G_j(\mathbf{v}, q) \quad \forall (\mathbf{v}, q) \in V_j.
$$

$G_j$ is a linear functional on $V_j$. In particular on the finest level, $G_j = (\mathbf{f}, \mathbf{v})$. Problem (6) can be written in matrix-vector form as $A_j x = b$ with a symmetric, indefinite matrix $A_j$. In the fine level smoothing of the multigrid method defined below, $m$ steps of a Jacobi-like relaxation are applied to the squared system $A_j x = A_j b$. The relaxation parameter $\omega_j$ below has to be less than or equal to the reciprocal of the spectral radius of $A_j$ (cf. [1, 12]).

As the multilevel spaces $\{ V_j \}$ are not nested, it is necessary to introduce an intergrid transfer operator $I_j : V_{j-1} \rightarrow V_j$. For simplicity in analysis and in implementation (other
intergrid transfer operators would lead working algorithms too, we define

\( I_j = \Pi_{T_j} \times \mathbf{P}_{T_j} \), \( I_j[u, p] = [\Pi_{T_j} u, \mathbf{P}_{T_j} p] \in V_j \)

for any \( u \in C(\Omega)^3 \) and \( p \in L^2(\Omega) \). Here in (7), we used the common notations that \( \Pi_{T_j} \) and \( \mathbf{P}_{T_j} \) are the nodal value interpolation operator and the \( L^2(\Omega) \) projection operator, respectively. We note that the computation of \( \mathbf{P}_{T_j} : P^0_{k,T_j} \rightarrow P^0_{k,T_j} \) is done element-wise since the pressure functions are discontinuous. Because the grids \( \{T_j\} \) are constructed on \( \{M_j\} \), we can see that

\[ V_j \cap V_{j+1} = (P^0_{k+1,M_j})^3 \times P^0_{k,M_j} =: \tilde{V}_j, \]

the space of piecewise continuous and discontinuous polynomials on grid \( M_j \).

**Definition 3.1** (Algorithm 3.1 in [12].)

1. **Smoothing.** Let \([u^0_j, p^0_j] \in V_j \) be a given guess to the solution of Problem (6). For \( l = 1, 2, \ldots, m \), compute the solutions of

\[
\begin{align*}
\left( w^j_l, v \right) + h^2_j(r^j_l, q) &= \sqrt{2} \left\{ G_j(v, q) - L\left([u^j_{l-1}, p^j_{l-1}], [v, q]\right) \right\} \quad \forall [v, q] \in V_j \\
(u^j_l - u^j_{l-1}, v) + h^2_j(p^j_l - p^j_{l-1}, q) &= L\left([w^j_l, p^j_l], [v, q]\right) \quad \forall [v, q] \in V_j.
\end{align*}
\]

2. **Correction.** Let \([u^*_j, p^*_j] \in V_{j-1} \) be the exact solution of Problem (6) with

\[
G_{j-1}(v, q) := G_j(I_j[v, q]) - L\left([u^m_j, p^m_j], I_j[v, q]\right) \quad \forall [v, q] \in V_j.
\]

If \( j = 0 \), put \([\hat{u}_j, \hat{p}_j] := [u^*_j, p^*_j] \).

If \( j > 0 \), compute an approximation \([\hat{u}_j, \hat{p}_j] \) by applying \( \alpha (\geq 2) \) iterations of the \((j - 1)\)-level scheme to (6) with starting value zero. Put

\[
[u^{n+1}_j, p^{n+1}_j] = [u^m_j, p^m_j] + I_j[\hat{u}_j, \hat{p}_j].
\]

In real computation the \( L^2 \) inner products in (9)–(10) are replaced by equivalent discrete \( L^2 \) inner products. This does not affect the analysis (cf. [1]).

4 **Convergence analysis**

In this section, we prove the constant rate of convergence for the multigrid algorithm defined in Definition 3.1. Let \( \{[\phi^j_t, \psi^j_t]\}^{\dim(V_j)}_{t=1} \) be the complete set of eigenfunctions for the symmetric bilinear functional \( L(\cdot, \cdot) \):

\[
L([\phi^j_t, \psi^j_t], [v, q]) = \lambda^j_t ([\phi^j_t, v] + h^2_j(\psi^j_t, q)).
\]

We can assume the eigenfunctions are normalized and that

\( 0 < |\lambda^j_t| \leq \cdots \leq |\lambda^j_{\dim(V_j)}| \).

Given \([u_j, p_j] = \sum_t c_j [\phi^j_t, \psi^j_t] \in V_j \), we define the \( \|\cdot\|_s \) norm by

\[
\| [u_j, p_j] \|_s := \left( \sum_{t=1}^{\dim(V_j)} c_j^2 |\lambda^j_t|^s \right)^{1/2}.
\]

We note that \( \| \cdot \|_0 \) is defined for all functions in \( H^1(\Omega)^3 \times L^2(\Omega) \) while the other norms \( \| \cdot \|_s \) are defined only in \( V_j \).

Let \([u^*_j, p^*_j] \in V_j \) denote the solution of Problem (6) and

\( [e^*_j, \epsilon^*_j] := [u^*_j - u^0_j, p^*_j - p^0_j] \)

be the error of the \( t \)-th iterate, \( 0 \leq t \leq m+1 \). The following fine-level smoothing property is shown in [12] under the conditions (4) and (2).

\[
\| [e^*_j, \epsilon^*_j] \|_2 \leq Ch_j^{-2} m^{-1/2} \| [u^0_j, p^0_j] \|_0.
\]

**Lemma 4.1** Let \([u_j, p_j] \in V_j \) be \( L^2 \) orthogonal to \( \tilde{V}_{j-1} \) in the sense that

\[
\langle u_j, v \rangle + h^2_j(p_j, q) = 0 \quad \forall [v, q] \in \tilde{V}_{j-1}.
\]

Then

\[
\| [u_j, p_j] \|_{-2} \leq Ch_j^2 \| [u_j, p_j] \|_0.
\]

**Proof** This lemma is almost identical to Lemma 4.2 in [12]. We note that the space \( \tilde{V}_{j-1} \) has the same order of approximation as that for \( V_{j-1} \) and that \( \tilde{V}_{j-1} \subset V_j \). Therefore the proof there remains the same.

**Theorem 4.1** Let \( \delta_{j,m} \) be the convergence rate measured in the \( \| \cdot \|_0 \)-norm of one iteration of multigrid algorithm defined in Definition 3.1 at level \( j \) with \( m \) relaxations. For every \( \kappa \in (0, 4^{-1/(\alpha-1)}) \) there is a number \( m_\kappa \), which depends on \( k \) and \( \kappa \), but not on \( j \), such that

\[
\delta_{j,m} \leq \kappa \quad \forall j, m \geq m_\kappa.
\]

**Proof** Following the frameworks of Verfürth [12] and Bank-Dupont [1], we need to prove the case of two-level multigrid algorithm, i.e., \([u^*_{j-1}, p^*_{j-1}] = [u^*_{j-1}, p^*_{j-1}] \) in (12). Let \([u^*_{j-1}, p^*_{j-1}] \) be the orthogonal projection of the
iterative error \( \|e_j^{m}, e_j^{m+1}\| \) in \( \widetilde{V}_{j-1} \). Since \( \widetilde{V}_{j-1} \subset V_{j-1} \), by (4), it follows by a duality argument (cf. [12] and [1]) that
\[
\|e_j^{m}, e_j^{m+1}\|_0 \\
\leq C h_j \inf_{v \in L^2(\widetilde{V}_{j-1})} \{ \|e_j^m - v\|_1 + \|e_j^m - q\|_0 \} \\
\leq C h_j \{ \|e_j^m - w_{j-1}\|_1 + \|e_j^m - r_{j-1}\|_0 \} \\
\leq C \|e_j^m, e_j^{m+1}\|_0 - \|w_{j-1}, r_{j-1}\|_0,
\]
where in the last step we used an inverse inequality (cf. [4]). Because of the nonnestedness of spaces, the \( m + 1 \)-st iterative error is actually
\[
\|e_j^{m+1}, e_j^{m+1}\|_0 \\
\leq \|e_j^m, e_j^m - I_j [u_j^m, p_j^m]\|_0 \\
+ \|I_j [u_j^m, p_j^m] - I_j [w_{j-1}, r_{j-1}]\|_0 \\
\leq C \|e_j^m, e_j^m\|_2 - \|w_{j-1}, r_{j-1}\|_0,
\]
where in the last step we used the \( L^2 \) stability of nodal-value interpolation operator when restricted on \( V_{j-1} \) (see [10] for a proof, where the averaging interpolation operator can be designed to be identical to \( \Pi_T \), when restricted to \( V_{j-1} \). Combining above two estimates we obtain
\[
\|e_j^{m+1}, e_j^{m+1}\|_0 \leq C \|e_j^m, e_j^m - [w_{j-1}, r_{j-1}]\|_0.
\]
Now, applying Lemma 4.1 and (15) we can get (cf. [12]) that
\[
\|e_j^m, e_j^m - [w_{j-1}, r_{j-1}]\|_0^2 \\
\leq C \|e_j^m, e_j^m\|_2 \|e_j^m - w_{j-1}, r_{j-1}\|_{-2} \\
\leq C \|e_j^m, e_j^m\|_2 \|e_j^m, e_j^m\|_2 - \|w_{j-1}, r_{j-1}\|_0,
\]
Therefore, the proof is completed by choose \( m_\kappa \) large enough as
\[
\|e_j^{m+1}, e_j^{m+1}\|_0 \leq C \|e_j^m, e_j^m\|_0.
\]

5 A combined conjugate gradient multigrid algorithm

In this section we define the second algorithm which is also based on an algorithm of Verfürth [13]. We show that the algorithm retains the constant rate of convergence even the multilevel spaces are nonnested in the present case. First we define an operator \( L : P^0_{k+1} \rightarrow P^0_k \) as follows. Given \( p \in P^0_k \), let \( u_k \in (P^0_{k+1})^3 \) and \( Lp \in P^0_k \) be the unique solutions of the equations
\[
a(u_k, v) = b(v, p) \quad \forall v \in (P^0_{k+1})^3,
\]
We remark that solving two linear systems is required for each evaluation of \( L \). The first system consists of three discrete Laplace equations for continuous piecewise polynomials, where we apply the multigrid method. The second linear system is a discrete mass equation which are uncoupled on each tetrahedron. This system can be solved locally with a cost proportional to the number of unknowns \( k \) fixed. Next let \( u_j \in (P^0_{k+1})^3 \) and \( g \in P^0_k \) be the unique solutions of the equations
\[
a(u_j, v) = f(v) \quad \forall v \in (P^0_{k+1})^3,
\]
\[
(b, q) = b(u_j, q) \quad \forall q \in P^0_k,
\]
It is shown in [13] that the pair \( [u, p] \in V_j \) is the solution of (3) on the top level if and only if
\[
Lp = g,
\]
where \( a(u, v) = (f, v) - b(v, p) \), for all \( v \in (P^0_{k+1})^3 \). We can write the method (20–24) in operator forms:
\[
L = BA^{-1}B^*, \quad g = BA^{-1}P_T f,
\]
where the operators are defined by \( a(A_j^{-1}u, v) := (u, v) \) (recall the notation \( A_j := A \) ), \( (Bu, q) := b(u, q) \) and \( (B^*_p, v) := b(v, p) \), for all \( v, q \in V_j \). For simplicity, we drop the index \( j \) if \( j = J \) and if there is no confusion. Here, \( P_T \) is also used to denote the \( L^2 \) orthogonal projection operator on the space \( (P^0_{k+1})^3 \). We let \( \omega_j \) denote the reciprocal of the spectral radius of \( A_j \). We now define a sequence of symmetric operators to approximate \( A_j^{-1} \) by the multigrid method (cf. [3]).

**Definition 5.1**

Set \( K_{0,n} = A_0^{-1}, \quad n = 1, 2, \ldots \).

Assume that \( K_{j, n} \) has been defined, define \( K_{j, 1} \) for \( z \in (P^0_{k+1})^3 \), as follows:

1. Set \( w^0 = 0 \) and \( e^0 = 0 \).
2. Define \( w^l \) for \( l = 1, \ldots, m \) by
\[
w^l = w^{l-1} + \omega_j (z - A_j w^{l-1}).
\]
3. Define \( w^{m+1} = w^m + \Pi_T e^\alpha, \alpha \geq 2, \) where \( e^l \) for \( l = 1, \ldots, \alpha \) is defined by
\[
e^l = e^{l-1} + K_{j-1, 1} (z - A_j w^m - A_{j-1} e^{l-1}).
\]
4. Define $w^l$ for $l = m + 2, \cdots, 2m + 1$ by (26).
5. Set $K_J z = w^{2m+1}$.

Finally, $K_{j,n} z$ is defined by doing $n$ cycles of above iterations:

\[(28) \quad K_{j,n} z = K_{j,n-1} z + K_{j,1} (z - A_{j} K_{j,n-1} z) \]

\[\Box\]

**Theorem 5.1** Let (4) hold. For any $0 < \delta < 1$, there is an integer $m$ independent of $j$, but depending on $\alpha$ and $k$, such that

\[(29) \quad \| (A_j^{-1} - K_{j,n}) z \|_1 \leq \delta^n \| z \|_1 \quad \forall z \in (P_{k+1,T_j})^3.\]

**Proof** This theorem is a corollary of Theorem 4.1 in [9], which proves the convergence of W-cycle nonnested multigrid methods. \[\Box\]

We remark that the multilevel spaces

\[(30) \quad (P_{k+1,T_j})^3 \not\subset (P_{k+1,T_j})^3 \not\subset \cdots \not\subset (P_{k+1,T_j})^3,\]

are used in our algorithm. This results in a nonnested multigrid method. One could use the following multilevel spaces

\[(31) \quad (P_{k+1,M_0})^3 \subset \cdots \subset (P_{k+1,M_1})^3 \subset (P_{k+1,T_j})^3\]

from our theory in the last iteration. The advantage of the latter is that one gets better convergence rate and less computational work inside each iteration (the convergence theory in this case is standard and covered by [1]). The disadvantage is that one has to set up two data structures to handle two different families of multilevel finite element spaces.

We now define a conjugate gradient algorithm for solving Problem (24), which is an equivalent problem of the Stokes equations (3). When we solve (24), we approximate $L_p$ for $p \in P_{k,T_j}$ by

$L_n p := B K_{f,n} B^* p$.

**Definition 5.2** (Algorithm 5.1 in [13])

1. Pre-processing: Compute

\[g^* := B K_{f,n} P_{T_j} f.\]

2. Start: Given an initial guess $p^0 \in P_{k,T_j}$ for the pressure $p_J$ solving (3). Compute

\[q^0 = L_n p^0\]

and put

\[r^0 = q^0 - g^*, \quad d^0 = -r^0.\]

Set $i = 0$, and $\epsilon$ to a small positive tolerance.

3. Iteration step: If $\|r^i\|_0 \leq \epsilon$ go to step 4. Otherwise compute

\[q^{i+1} = L_n d^i\]

and set

\[\alpha^{i+1} = -\frac{(r^i, d^i)}{(q^{i+1}, d^i)}, \quad p^{i+1} = p^i + \alpha^{i+1} d^i, \quad r^{i+1} = r^i + \alpha^{i+1} q^{i+1}, \quad \beta^{i+1} = \frac{(r^{i+1}, r^{i+1})}{(r^i, r^i)}, \quad d^{i+1} = -r^{i+1} + \beta^{i+1} d^i.\]

Replace $i$ by $i + 1$ and return to the beginning of the this step.

4. Post-processing: Compute

\[u^i := K_{J,n} (P_{T_j} f - B^* p^i)\]

and take $[u^i, p^i] \in V_J$ as the final approximation to the solution of (3).

\[\Box\]

**Theorem 5.2** Let $[u^i, p^i]$ be defined in Definition 5.2 and $[u_J, p_J]$ the solution of (3). Then

\[\| u_J - u^i \|_1 + \| p_J - p^i \|_0 \leq \frac{1}{C - \delta^n} \{ \epsilon C + \delta^n C \| f \|_0 + \delta^n C \| p_J \|_0 \},\]

where $\epsilon$ is defined in Definition 5.2 and $\delta$ is defined in Theorem 5.1.

**Proof** This is Proposition 5.1 in [13]. \[\Box\]

**6 A numerical test**

We test the combined conjugate gradient-multigrid method defined in Definition 5.1 in solving a model 2D Stokes problem on a unit right triangle (mesh for $M_0$). We use 2D mixed elements on macro-element meshes where each triangle of $M_j$ is subdivided into three triangles when generating $T_j$, similarly to the 3D case depicted by Figure 1. For $k = 1$ and $k = 2$, such mixed elements have been shown stable in [7]. The analysis for the multigrid methods provided in this manuscript remains the same for the 2D case. But we should point out that for high degree ($k > 2$) polynomials such macro-element meshes have
no advantage as the regular meshes would provide stable mixed elements ([8]). The table in Figure 2 lists the numbers of the outer conjugate iterations when applying the algorithm of Definition 5.2 where $K_{J,n} = A^{-1}$. Also the condition numbers of operator $BA^{-1}B^*$ in the $L^2$ inner product are listed in Figure 2. We remark that in the conjugate iteration, we have to use the $L^2$ mass matrix for pressure functions as a preconditioner. One can find such a preconditioned conjugate iteration in [14]. One can see from the numerical data that the operator $BA^{-1}B^*$ is well conditioned, independently of the polynomial degree $k$. Numerical data also indicate that $BA^{-1}B^*$ remains well conditioned when we refine meshes. For example, for $k = 1$, $C(BA^{-1}B^*) = 18.8885$ and $19.3166$ on $T_2$ and $T_3$, respectively.

However, when $A^{-1}$ is replaced by the multigrid approximation $K_{J,n}$, the number of conjugate iterations and the condition number of $BK_{J,n}B^*$ both increase with the polynomial degree $k$ and the mesh level $J$. This can be observed by the data listed in the table in Figure 3. Here, we apply the two-level nonnested multigrid method defined in Definition 5.1 where $m = 4$ (doing 4 pre-smoothings and 4 post-smoothings), and $n = 4$ (doing 4 cycles of multigrid iterations). When we increase $m$, or $n$, or both to get a better approximation $K_{J,n}$ for $A^{-1}$, the data in Figure 3 will approach those listed in Figure 2. Unlike the case of $BA^{-1}B^*$, if we fix the polynomial degree and refine the mesh, the condition number of $BK_{J,n}B^*$ would become worse. For example, using quadratic polynomials for the velocity, $C(BK_{J,n}B^*) = 25.7506$ and $70.7792$ on $T_2$ and $T_3$, respectively.

References


