On the jump-independence of the weighted $L^2$ projection

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Abstract

This work gives an affirmative answer to a long time research problem. The weighted $L^2$ projection is of optimal order in approximation measured in the weighted $L^2$ norm, and stable in the weighted semi-$H^1$ norm, independent of the jump of the constant weights, in 2D and 3D finite element spaces.

Keywords. jump coefficients, finite element, $L^2$ projection, weighted projection.


1 Introduction

When studying the numerical solution of elliptic boundary value problems with discontinuous coefficients, a useful tool is the weighted $L^2$ projection operator, cf. [2, 7, 9]. To be specific, we consider a domain $\Omega$ which is subdivided into finite many, bounded, polygonal subdomains $\{\Omega_i\}$, in $d = 2$ or 3 space-dimension. On each subdomain $\Omega_i$, we are given a positive constant $\omega_i$, and we have a quasi-uniform triangulation of size $h$, cf. [3, 4], shown by Figure 1 as an example. Thus each $\Omega_i$ and $\Omega$ is Lipschitz. We further assume the subdomain grids are matching so that we define a continuous piecewise linear finite element space on the combined grid $T_h$ over the domain $\Omega$, [4]:

$$V_h = \{ v \in H_0^1(\Omega) \cap C(\Omega) \mid v|_K \in P_1 \quad \forall K \in T_h \}. \quad (1.1)$$

We limit ourselves to this $P_1$ conforming element though the analysis would remain valid for high order elements.

Figure 1: Constant weight each subdomain, and a matching grid.

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The weighed \( L^2 \) and semi-\( H^1 \) inner products are defined by

\[
(u, v)_{L^2_\omega(\Omega)} = \sum_{i=1}^{J} \omega_i \int_{\Omega_i} u v \, dx, \quad (1.2)
\]

\[
(u, v)_{H^1_\omega(\Omega)} = \sum_{i=1}^{J} \omega_i \int_{\Omega_i} \nabla u \cdot \nabla v \, dx. \quad (1.3)
\]

Conventionally, we use \( H^1_\omega(\Omega) \) for \( L^2_\omega(\Omega) \) sometimes in this manuscript. The induced norms are denoted by \( \| \cdot \|_{L^2_\omega} \) and \( | \cdot |_{H^1_\omega} \), respectively. The weighted \( L^2 \) projection \( Q^\omega_h : L^2(\Omega) \mapsto V_h \) is defined by

\[
(Q^\omega_h u, v)_{L^2_\omega(\Omega)} = (u, v)_{L^2_\omega(\Omega)} \quad \forall v \in V_h. \quad (1.4)
\]

The following important theorem is proved in [2], about 20 years ago:

**Theorem 1.1** ([2]) If for all \( i \), the \((d-1)\)-dimensional Lebesgue measure of \( \partial \Omega_i \cap \partial \Omega \) is positive, then for all \( u \in H^1_0(\Omega) \)

\[
\| u - Q^\omega_h u \|_{L^2_\omega(\Omega)} + h |Q^\omega_h u|_{H^1_\omega(\Omega)} \leq C h |\log h|^{1/2} |u|_{H^1_\omega(\Omega)}, \quad (1.5)
\]

where \( C \) is independent of \( \{ \omega_i \} \).

Trying to show the necessity that all subdomains touch the boundary \( \partial \Omega \), and of the log term in the bound, several examples are constructed in [8]. Nevertheless, those examples only show the Poincaré inequality fails to hold independent of the weight jump. In particular, see also (4.9) and (4.10) and the third numerical test in Section 4, for \( u \in H^1_0(\Omega) \),

\[
\| Q^\omega_h u \|_{L^2_\omega(\Omega)} \not\leq C |u|_{H^1_\omega(\Omega)}. \]

In fact, we will lift the touching boundary requirement, and the log term as well, off of Theorem 1.1, in this work, giving the following theorem on the optimal order approximation and the stability of weighted \( L^2 \) projection. In Section 2, we introduce and modify the Scott-Zhang interpolation operator. In Section 3, we extend the Bramble-Hilbert lemma to non-Lipschitz subdomains, and prove the main theorem. Several numerical tests, including some worst cases, are given in Section 4, supporting the new theory.

**Theorem 1.2** Let the weighted \( L^2 \)-projection operator \( Q^\omega_h \) be defined in (1.4). For all \( u \in H^1_0(\Omega) \),

\[
\| u - Q^\omega_h u \|_{L^2_\omega(\Omega)} + h |Q^\omega_h u|_{H^1_\omega(\Omega)} \leq C h |u|_{H^1_\omega(\Omega)}, \quad (1.6)
\]

where \( C \) is independent of \( \{ \omega_i \} \).

2 The Scott-Zhang operator

The main work is on the analysis of Scott-Zhang operator, \( \Pi_h : H^1_0(\Omega) \mapsto V_h \), cf. [6]. The operator is a local averaging operator. Though the operator \( \Pi_h \) is linear, it is not uniquely defined. In the application here, we need to make an appropriate selection of \((d-1)\)-dimensional simplex \( \sigma_x \), in the averaging, for each vertex \( x \) in the triangulation.
If a vertex \( x \) is on the boundary \( \partial \Omega_h \), we choose any one boundary triangle (\( d = 3 \)) or one boundary edge (\( d = 2 \)), on which the vertex \( x \) is. This is to ensure \( \Pi_h v(x) = 0 \) for all \( x \in \partial \Omega \), to preserve the homogeneous boundary condition of \( V_h \).

If \( x \) is an internal vertex of \( T_h \), there are two situations. If \( x \) is only on one subdomain \( \Omega_i \), we pick up any triangle or edge as \( \sigma_x \) from \( \Omega_i \). If \( x \) is shared by more than one \( \Omega_i \), we select one triangle (3D) or one edge from the \( \Omega_x \) on which the \( \omega_i \) is biggest. That is, we select

\[
x \in \sigma_x \subset K \in T_h(\Omega_i), \text{ such that } \omega_i = \max\{\omega_j \mid x \in \Omega_j\}.
\]

We note that the rule is applied to internal vertex only. At a boundary vertex, since we will have \( \Pi_h v(x) = 0 \), we choose any boundary (\( d - 1 \)) simplex.

On the (\( d - 1 \))D reference element \( \sigma \), either the unit triangle \( 0 \leq x, y, 1 - x - y \leq 1 \) or the unit line \([0, 1]\), we have a dual basis \( \{\phi_i\} \) for \( P_1 \) nodal basis there:

\[
\int_\sigma \phi_j \hat{\phi}_i dx = \delta_{ij}.
\]

The Scott-Zhang interpolation is defined by

\[
\Pi_h : H^1_0(\Omega) \to V_h, \quad \Pi_h v(x) = \sum_{x_i \in T_h} \phi_{x_i}(x) \int_{\sigma_{x_i}} \psi_{x_i}(y) v(y) dy,
\]

where \( \phi_{x_i} \) is the \( P_1 \) hat basis function at point \( x_i \), and \( \psi_{x_i} = \det(B_{x_i})^{-1}(\hat{\psi}_{x_i} \circ F_{x_i}^{-1}) \). Here \( K \) is the element in \( T_h \) on which \( \sigma_{x_i} \) is a face (\( d - 1 \)) simplex, and \( F_{x_i}(\hat{x}) = x_i + B_{x_i} \hat{x} \) is the affine mapping from the reference \( \hat{K} \) to \( K \). We use the restriction of \( F_{x_i}^{-1} \) on \( \sigma_{x_i} \) to define \( \det(B_{x_i}) \).

For details, please read [6]. First of all, as an interpolation operator, \( \Pi_h \) preserves the image space:

\[
\Pi_h v_h = v_h \quad \forall v_h \in V_h.
\]

**Lemma 2.1** On any element \( K \in \Omega_i \subset \Omega \), for any \( v \in H^1(\Omega) \),

\[
\|\Pi_h v\|_{H^m(K)} \leq C(h^{-m}\|v\|_{L^2(S_K)} + h^{-m+1}\|v\|_{H^1(S_K)}), \quad m, 0, 1.
\]

Here \( S_K \) is a union of neighboring elements of \( K \), excluding those in lower index subdomains \( \Omega_k \), \( k < i \), cf. Figure 3:

\[
S_K = \bigcup\{K' \in T_h \mid \overline{K'} \cap \overline{K} \neq \emptyset, \text{ and } K' \not\subset \Omega_k \text{ for } k < i\}.
\]

For convenience, we assume here that

\[
\omega_k \leq \omega_i \quad \text{if } k < i.
\]
Corollary 2.1

For all $u \in H^1_0(\Omega)$,

$$
\| u - \Pi_h u \|_{L^2(\Omega_i)} \leq Ch| u |_{H^1(\bigcup_{k \leq i} \Omega_k)}.
$$

Corollary 2.1 *For all* $u \in H^1_0(\Omega)$,

$$
\| u - \Pi_h u \|_{L^2(\Omega_i)} \leq Ch| u |_{H^1(\bigcup_{k \geq i} \Omega_k)}.
$$

**Proof.** This is a simplified version of Theorem 3.1 in [6]. The only change is on the definition of $S_K$. Basically, it needs only the scaling argument, commonly used in the finite element analysis. But we repeat the proof here for the convenience of readers. Let the vertices of $K$ be $\{ x_i \}_{i=0}^d$. By the affine mappings and the trace theorem,

$$
\| \Pi_h v \|_{L^2(K)} \leq \sum_{i=0}^d \| \Pi_h v(x_i) \|_{L^2(K)} \leq Ch^{d/2} \max_{0 \leq i \leq d} \| \phi_i \|_{L^2(K)} \sum_{i=0}^d \int_{\sigma_i} \psi_i(x) v(x) dx
$$

$$
\leq Ch^{d/2} \sum_{i=0}^d \| \psi_i \|_{L^\infty(\sigma_i)} \| v \|_{L^1(\sigma_i)}
$$

$$
\leq Ch^{d/2} \sum_{i=0}^d \| \det(B_{\sigma_i})^{-1} \|_{L^\infty(\sigma)} \| v \|_{L^1(\sigma)}
$$

$$
\leq Ch^{d/2} \sum_{i=0}^d \| \hat{v} \|_{H^1(K_i)} = Ch^{d/2} \sum_{i=0}^d \left( h^{-d/2} \| v \|_{L^2(K_i)} + h^{1-d/2} \| v \|_{H^1(K_i)} \right)
$$

$$
\leq C \left( \| v \|_{L^2(S_K)} + h \| v \|_{H^1(S_K)} \right).
$$

Here in the last step, we used the definition of $S_K$ which includes all elements $K'$ on which $\omega_k$ is bigger, cf. Figure 3. Similarly, we get

$$
\| \Pi_h v \|_{H^1(K)} \leq \sum_{i=0}^d \| \Pi_h v(x_i) \|_{H^1(K)} \leq C h^{d/2-1} \sum_{i=0}^d \| \Pi_h v(x_i) \|
$$

$$
\leq C h^{d/2-1} \sum_{i=0}^d \left( h^{-d/2} \| v \|_{L^2(K_i)} + h^{1-d/2} \| v \|_{H^1(K_i)} \right)
$$

$$
\leq C \left( h^{-1} \| v \|_{L^2(S_K)} + \| v \|_{H^1(S_K)} \right).
$$

In particular, as we avoid $K'$ of lower index $\Omega_k$ in defining $S_K$, we have the following corollary, but we may have

$$
\| u - \Pi_h u \|_{L^2(\Omega_i)} \leq Ch| u |_{H^1(\bigcup_{k \leq i} \Omega_k)}.
$$

**Corollary 2.1** For all $u \in H^1_0(\Omega)$,

$$
\| u - \Pi_h u \|_{L^2(\Omega_i)} \leq Ch| u |_{H^1(\bigcup_{k \geq i} \Omega_k)}.
$$


Figure 4: $S_K$ is not a star-shaped domain, neither a union of star-shaped domains.

The standard proof next would be applying the Bramble-Hilbert lemma, showing the equivalence of $H^1$-semi norm and the polynomial-modulus norm, cf. [1, 4]) and Lemma 3.1 below. However, different from Scott-Zhang [6], the domain $S_K$ is no longer a star-shaped one, cf. Figure 4, and we could not apply directly the Bramble-Hilbert lemma. As the whole domain is Lipschitz, we extend the Bramble-Hilbert lemma to this special case of subdomain in next section and prove our main theorem.

3 A subdomain Bramble-Hilbert lemma

We modify the method of Dupont and Scott [5] to show the Bramble-Hilbert lemma holds on two touching triangles or tetrahedra. With the extended Bramble-Hilbert lemma, it is standard to show the optimal approximation of weighted $L^2$ projection.

**Lemma 3.1** Let $\Omega$ be a 2D or 3D Lipschitz domain, with a quasi-uniform triangulation $T_h$. Let $K$ and $K'$ be such that $K \neq K'$, $K \cap \overline{K'} \neq \emptyset$, cf. Figure 5, and $K \cap \overline{K'} \notin \partial \Omega$. Then for $f \in C_0^\infty(\Omega)$

$$\|f - Qf\|_{L^2(K)} \leq C h |f|_{H^1(K \cup K')},$$

(3.1)

where

$$Qf = \int_B f(x) \rho(x) dx$$

(3.2)

is a constant over $(K \cup K')$. Here $\rho(x) \in C_0^\infty(B)$ is a standard smoother ([5]) such that $\int_B \rho(x) dx = 1$, and $B \subset K'$ is a ball of radius $r = Ch$.

**Proof.** If $K = K'$, or $K$ and $K'$ share a common edge in 2D, or share a face-triangle in 3D, then $(K \cup K')$ is a star-shaped domain and the lemma is shown in [5]. Thus, we need to prove the lemma in the worst case, $K$ and $K'$ sharing one common vertex, which is included in all cases.

Let $f \in C_0^\infty(\Omega)$. Let $z \in (\overline{K} \cap \overline{K'})$, $z \notin \partial \Omega$. Let $x \in B \subset K'$ and $y \in (K \cup K')$, cf. Figure 5. To deal with differentiation at the turning point $z$, we define a corrected $C^\infty$ function:

$$\tilde{f}_x(y) = \begin{cases} f(y) & \text{if } y \in K', \\ f(y) + \alpha_{xy} |y - z| & \text{if } y \in K, \end{cases}$$
where \( \alpha_{xy} \) is the difference of two directional derivatives

\[
\alpha_{xy} = \nabla f(z) \cdot \left( \frac{z - x}{|z - x|} - \frac{y - z}{|y - z|} \right).
\]

Let us assume \( y \in K \) first. Then on the polyline \( xzy \),

\[
l(s) = \begin{cases} 
  x + ss_0^{-1}(z - x) & \text{if } 0 \leq s \leq s_0, \\
  z + (s - s_0)(1 - s_0)^{-1}(y - z) & \text{if } s_0 \leq s \leq 1,
\end{cases}
\]

where

\[
s_0 = \frac{|z - x|}{|z - x| + |y - z|}.
\]

For this \( C^\infty \) function on the polyline, we have

\[
\tilde{f}_x(y) = \tilde{f}_x(x) + \int_0^1 \frac{d}{ds} \tilde{f}_x(l(s))ds.
\]

That is,

\[
f(y) + \alpha_{xy}|y - z| = f(x) + \int_0^{s_0} s_0^{-1}\nabla f(l(s)) \cdot (z - x)ds
\]

\[
+ \int_{s_0}^1 (1 - s_0)^{-1} \nabla f(l(s)) \cdot (y - z) + \alpha_{xy}|y - z|)ds.
\]

Thus,

\[
f(y) = f(x) + \int_0^{s_0} s_0^{-1}\nabla f(l(s)) \cdot (z - x)ds + \int_{s_0}^1 (1 - s_0)^{-1}\nabla f(l(s)) \cdot (y - z)ds.
\]

In other words, the construction, which corrects \( f \) at the turning point \( z \), gives the same Taylor formula as if the three points \( x, y \) and \( z \) are on a same line, i.e.,

\[
f(y) = f(x) + \int_0^1 \frac{d}{ds} f(l(s))ds. \tag{3.3}
\]

Here in (3.3), \( y \) can be in either \( K \) or \( K' \).
For a fixed \( y \) in \( K' \), we integrate (3.3) against \( \rho(x) \) for \( x \) over \( B \), cf. Figure 5.

\[
f(y) = \int_B f(x)\rho(x)dx + \int_B \int_0^1 \frac{d}{ds}f(1(s))d\sigma(x)dx \\
= Qf + \int_B \int_0^1 \nabla f(1(s)) \cdot (y-x)ds\rho(x)dx.
\]  

(3.4)

By the Schwartz inequality,

\[
|f(y) - Qf|^2 \leq \int_B \int_0^1 (\nabla f(x+s(y-x)) \cdot (y-x))^2 d\sigma(x)dx \int_B \int_0^1 \rho(x)dsdx \\
= \int_B \int_0^1 (\nabla f(x+s(y-x)) \cdot (y-x))^2 \rho(x)dsdx.
\]  

(3.5)

Next, we change variables 

\[ w = x + s(y-x). \]

Let \( C_y \) be the cone formed by the point \( y \) and the ball \( B \), cf. Figure 6,

\[ C_y = \{ w \in K' \mid w = x + s(y-x), \quad \text{for all } x \in B \text{ and } 0 \leq s \leq 1 \}. \]

As \( x \in B \) where \( B \subset K' \) is ball of radius \( r \) centered at \( x_0 \), we have

\[
w \in C_y \Leftrightarrow |w - y + (1-s)(y-x_0)| = (1-s)|x-x_0| \leq (1-s)r \\
\Rightarrow |w - y| - (1-s)|y-x_0| \leq (1-s)r \\
\Leftrightarrow s \leq 1 - \frac{|w - y|}{|y-x_0| + r}.
\]

Figure 6: Cone \( C_y \), when \( y \) is inside or outside the ball \( B \).
With the new variables, (3.5) is transformed to

$$|f(y) - Qf|^2 \leq \int_{C_y} \int_0^{1 - \frac{|w-y|}{|y-x_0| + r}} \left( \nabla f(w) \cdot \frac{w-y}{s} \right)^2 \rho(\frac{w-sy}{1-s}) ds \frac{dw}{(1-s)^d}$$

$$\leq \|\rho\|_{L^\infty(B)} \int_{C_y} \int_0^{1 - \frac{|w-y|}{|y-x_0| + r}} \left( \nabla f(w) \cdot (w-y) \right)^2 ds \frac{dw}{(1-s)^d+2}$$

$$= \|\rho\|_{L^\infty(B)} \int_{C_y} (\nabla f(w) \cdot (w-y))^2 \left( \frac{(1-s)^{-d-1}}{d+1} \right) \frac{ds}{1 - \frac{|w-y|}{|y-x_0| + r}}$$

$$\leq Ch^{-d} \int_{C_y} \int_0^{1 - \frac{|w-y|}{|y-x_0| + r}} \frac{(1-s)^{-d-1}}{d+1} (|w-y|^d)^{-1} ds \frac{dw}{|w-y|^{d-1}}.$$

Here $\|\rho\|_{L^\infty(B)} = Ch^{-d}$, cf. [5], $|y-x_0| < h$ and $r < h$. As $f \in C^\infty_0(\Omega)$, we apply the Fubini’s theorem to get

$$\int_{K'} |f(y) - Qf|^2 dy \leq Ch \int_{K'} \int_{K_y} |\nabla f(w)|^2 \frac{dw}{|w-y|^{d-1}} dy$$

$$\leq Ch \int_{K'} \int_{K_y} |\nabla f(w)|^2 \frac{dw}{|w-y|^{d-1}} dy$$

$$= Ch \int_{K'} \int_{K_y} |\nabla f(w)|^2 \int_{K_y} \frac{dy}{|w-y|^{d-1}} dw. \quad (3.6)$$

Using the polar coordinates at the origin $w$ in 2D and 3D, cf. [3], we have

$$\int_{K_y} \frac{dy}{|w-y|^{d-1}} \leq C \max_{y \in K'} |w-y| = Ch.$$

Thus, (3.6) is simplified to

$$\|f - Qf\|_{L^2(K')}^2 \leq Ch^2 \int_{K'} |\nabla f(w)|^2 dw = Ch^2 |f|_{H^1(K')}^2. \quad (3.7)$$

Next, let $y \in K$. The integral in (3.3) would continue into the second element $K$. But the portion of path inside $K$ is independent of $x \in B$. Thus, (3.4) would be

$$f(y) - Qf = \int_B \int_0^{s_0} \nabla f(l(s)) \cdot \frac{z-x}{s_0} ds \rho(x) dx + \int_B \int_{s_0}^1 \nabla f(l(s)) \cdot \frac{y-z}{1-s_0} ds \rho(x) dx$$

$$= \int_B \int_0^{s_0} \nabla f(l(s)) \cdot \frac{z-x}{s_0} ds \rho(x) dx + \int_{s_0}^1 \nabla f(l(s)) \cdot \frac{y-z}{1-s_0} ds.$$

Hence,

$$|f(y) - Qf|^2 \leq 2 \left| \int_B \int_0^{s_0} \nabla f(l(s)) \cdot \frac{z-x}{s_0} ds \rho(x) dx \right|^2$$

$$+ 2 \left| \int_{s_0}^1 \nabla f(l(s)) \cdot \frac{y-z}{1-s_0} ds \right|^2. \quad (3.8)$$

$$+ 2 \left| \int_0^{s_0} \nabla f(l(s)) \cdot \frac{y-z}{1-s_0} ds \right|^2. \quad (3.9)$$
For the first term, we have

\[ \left\| \int_B \int_0^s \nabla f(l(s)) \cdot \frac{z-x}{s_0} ds \rho(x) dx \right\|^2 \leq \| \rho \|^2 \| \nabla f(l(s)) \|^2 \int_B \int_0^s \left| \nabla f(l(s)) \frac{h}{s_0} ds \right|^2 dx \]

\[ \leq C h^2 - 2d \int_{K'} |\nabla f(w)| dw \]

\[ \leq C h^2 - 2d \int_{K'} |\nabla f(w)|^2 dw \int_{K'} dw \]

\[ \leq C h^2 - d |f|_{H^1(K')}^2. \]

Integrating (3.9) over K, we have

\[ \| f - Qf \|^2 \|_{L^2(K)} \leq C h^2 |f|_{H^1(K')} + C \frac{h^2}{1 - s_0} \int_{K'} |\nabla f(y)|^2 dy ds \]

Therefore (3.1) holds for all \( C_0^\infty(\Omega) \) functions. The lemma is proven.

**Corollary 3.1** For all \( v \in H_0^1(\Omega) \) and all \( K \) and \( K' \) defined in Lemma 3.1,

\[ \| v - Qv \|^2 \|_{L^2(K)} \leq C h |v|_{H^1(K \cup K')} \]

**Proof.** Let \( \epsilon > 0 \). As \( C_0^\infty(\Omega) \) is dense in \( H_0^1(\Omega) \), and \( L^1(\Omega) \subset L^2(\Omega) \), we let \( f \in C_0^\infty(\Omega) \) such that

\[ \| v - f \|^2 \|_{H^1(\Omega)} \leq \epsilon, \quad \text{and} \quad \| v - f \|^2 \|_{L^1(\Omega)} \leq \epsilon. \]

Because \( Qv \) remains valid for \( v \in H_0^1(\Omega) \), by (3.1), we get

\[ \| v - Qv \|^2 \|_{L^2(K)} \leq \| v - f \|^2 \|_{L^2(K)} + \| f - Qf \|_{L^2(K)} + \| Q(f - v) \|_{L^2(K)} \]

\[ \leq \epsilon + C h |f|_{H^1(K \cup K')} + |K| \int_{B \subset K'} |f - v| \rho(x) dx \]

\[ \leq \epsilon + C h (|f - v|_{H^1(K \cup K')} + |v|_{H^1(K \cup K')} + |K||f - v|_{L^1(K')} \]

\[ \leq \epsilon(1 + C h + |K|) + C h |v|_{H^1(K \cup K')} \]

Let \( \epsilon \to 0 \). The proof is completed.

The \( S_K \) defined in (2.4) can be much bigger, as we casually include all elements sharing a vertex with \( K \), if those elements have larger coefficient \( \omega_i \) than that on \( K \). But the worst case is depicted in Figure 7, i.e., \( S_K \) has \( (d + 1) \) “isolated” elements, sharing the \((d + 1)\) vertices of \( K \).

**Theorem 3.1** For all \( v \in H_0^1(\Omega) \),

\[ \| v - \Pi_h v \|^2 \|_{L^2(K)} \leq C h |v|_{H^1(S_K)}, \]

(3.10)

\[ |\Pi_h v|_{H^1(K)} \leq C |v|_{H^1(S_K)}. \]

(3.11)
Figure 7: On $K'$, the $\omega_i$ is the largest among all $\omega_j$ on $S_K$, cf. (2.4).

**Proof.** Let $K' \in S_K$ be the one has the largest coefficient, i.e.,

$$K' \subset \Omega_i, \quad S_K \setminus K' \subset \bigcap_{j=1}^l \Omega_j.$$ 

By the extended Bramble-Hilbert lemma (3.1) or corollary 3.1, (if $K' = K$ (3.7) would be used as a trivial case,) we have

$$\|v - Qv\|_{L^2(S_K)} \leq C h |v|_{H^1(S_K)},$$

where $Qv$ is the average of $v$ on $K'$ only, defined in (3.2). We note that the line integral (3.3) would continue into every element of $S_K$, cf. Figure 7. By (2.3), thus,

$$\|v - \Pi_h v\|_{L^2(K)} \leq \|v - Qv\|_{L^2(K)} + \|\Pi_h (v - Qv)\|_{L^2(K)}$$

$$\leq \|v - Qv\|_{L^2(K)} + C (\|v - Qv\|_{L^2(S_K)} + h |v - Qv|_{H^1(S_K)})$$

$$\leq C h |v|_{H^1(K \cup K')} + C (C h |v|_{H^1(S_K)} + h |v|_{H^1(S_K)})$$

$$\leq C h |v|_{H^1(S_K)}.$$

By the same reasoning, i.e., (2.3) again, we have

$$|\Pi_h v|_{H^1(K)} = |\Pi_h (v - Qv)|_{H^1(K)} \leq C h^{-1} |v - Qv|_{L^2(S_K)} + |v - Qv|_{H^1(S_K)}$$

$$\leq C |v|_{H^1(S_K)}.$$

We come to prove the main theorem of this research.

**Proof.** (Theorem 1.2.) Noting the orthogonal projection property of $L^2_\omega$, we would get, by (3.10),

$$\|u - Q_h u\|_{L^2_\omega(\Omega)}^2 \leq \|u - \Pi_h u\|_{L^2(\Omega)}^2 = \sum_i \sum_{K \subset \Omega_i} \omega_i \|u - \Pi_h u\|_{L^2(K)}^2$$

$$= C h^2 \sum_i \sum_{K \subset \Omega_i} \omega_i |u|_{H^1(S_K)}^2 \leq C h^2 M_0 \sum_{K \subset \Omega_i} \omega_i |u|_{H^1(K)}^2$$

$$= C h^2 |u|_{H^1(\Omega)}^2.$$
where $M_0$ is the maximum of the number of elements meeting at a vertex of the triangulation, $x \in T_h$. Again, in order to have the constant $C$ above independent of $\omega_i$, we have to avoid the neighboring element $K'$ of lower index subdomain, when selecting $\sigma_x$ and $S_{K'}$. Similarly we get the stability result:

$$\|Q_\omega^h u\|_{H^2(\Omega)}^2 \leq 2\|Q_\omega^h u - \Pi_h u\|_{H^1(\Omega)}^2 + 2\|\Pi_h u\|_{H^1(\Omega)}^2$$

$$\leq 2 \sum_i \sum_{K \subset \Omega_i} \omega_i \left( Ch^{-2} \|Q_\omega^h u - \Pi_h u\|_{L^2(K)}^2 + \|\Pi_h u\|_{H^1(K)}^2 \right)$$

$$\leq C \sum_i \sum_{K \subset \Omega_i} \omega_i \left( h^{-2} \|Q_\omega^h u - u\|_{L^2(K)}^2 + 2\|\Pi_h u - u\|_{H^1(K)}^2 \right)$$

$$\leq Ch^{-2} \|Q_\omega^h u - u\|_{L^2(\Omega)}^2 + C\|u\|_{H^1(\Omega)}^2 \leq C\|u\|_{H^1(\Omega)}^2.$$ 

Here we used the inverse inequality on the finite element function $(Q_\omega^h u - \Pi_h u)$. Note that the inverse inequality comes from the scaling technique, which does not require function $(Q_\omega^h u - \Pi_h u)$ small, unlike the “forward” inequality which needs so.

### 4 Numerical tests

In the first test, the function $u$ to be projected is defined on the unit square, by

$$u(x,y) = 4x(1-x)y(1-y) \quad (x,y) \in \Omega = [0,1]^2. \quad (4.1)$$

The weights are defined by, shown in Figure 8,

$$\omega = \begin{cases} \epsilon & \text{if } (x,y) \in [.25,.75] \times [.25,.75], \\ 1 & \text{elsewhere on } \Omega. \end{cases} \quad (4.2)$$

We connect the $45^0$ diagonal of $\Omega$ as the level 1 grid and refine each triangle into 4 to define next level grid.

![Figure 8: Constant weights in (4.2), and a level 4 grid.](image)

The weighted $L^2_\omega$ projection of $u$ in to $V_h$ on each grid, defined by (1.4), is computed. We also compute the $L^2_\omega$ error and the semi-$H^1_\omega$ norm for the solution. In addition, because the
order of convergence is 2 for the $L^2$ projection, we compute the ratio

$$C_1 = \frac{h^2|u|_{H^1}}{\|u - Q^e_h u\|_{L^2}},$$

(4.3)

instead of $h|u|_{H^1}/\|u - Q^e_h u\|_{L^2}$. Usually, one computes the meaningful ratio $h^2|u|_{H^2}/\|u - Q^e_h u\|_{L^2}$. To see the stability, we compute the ratio

$$C_2 = \frac{|Q^e_h u|_{H^1}}{|u|_{H^1}}.$$  

(4.4)

The results are listed in Table 1. Here we use the conjugate gradient method to solve the positive definite linear system of the $L^2$ projection. As in the standard $L^2$ projection, the number of different eigenvalues of the mass matrix is finite, independent of higher refinement. This can be seen in the last column of Table 1. From Table 1, it is apparent that the operator $Q^e_h$ behaves the same way as the standard $L^2$ projection operator and the nodal value interpolation operator, such as the Scott-Zhang operator [6].

Table 1: The weighted $L^2$ projection in 2D function (4.1).

| $\|u - Q^e_h u\|_{L^2}$ | $h^n$ | $|Q^e_h u|_{H^1}$ | $C_1(4.3)$ | $C_2(4.4)$ | # CG |
|-------------------------|-------|------------------|-------------|-------------|------|
| $\epsilon = 10^{-6}$ in (4.2) |
| 3 | 0.040313 | 1.1 | 2.4867 | 3.4396 | 1.1209 | 12 |
| 4 | 0.011657 | 1.8 | 2.3206 | 3.0210 | 1.0296 | 57 |
| 5 | 0.003091 | 1.9 | 2.2785 | 2.8580 | 1.0076 | 206 |
| 6 | 0.000792 | 2.0 | 2.2676 | 2.7896 | 1.0019 | 347 |
| 7 | 0.000200 | 2.0 | 2.2648 | 2.7578 | 1.0005 | 349 |
| 8 | 0.000050 | 2.0 | 2.2641 | 2.7425 | 1.0001 | 340 |

| $\epsilon = 1$ in (4.2) |
| 3 | 0.054150 | 1.4 | 2.5667 | 2.6603 | 1.1136 | 8 |
| 4 | 0.015056 | 1.8 | 2.4338 | 2.4556 | 1.0286 | 17 |
| 5 | 0.003911 | 1.9 | 2.3975 | 2.3777 | 1.0072 | 32 |
| 6 | 0.000994 | 2.0 | 2.3882 | 2.3426 | 1.0018 | 32 |
| 7 | 0.000250 | 2.0 | 2.3859 | 2.3258 | 1.0004 | 29 |
| 8 | 0.000063 | 2.0 | 2.3853 | 2.3175 | 1.0000 | 26 |

| $\epsilon = 10^6$ in (4.2) |
| 3 | 3.7504658 | 1.7 | 648.829 | 1.042 | 1.038 | 15 |
| 4 | 9.541448 | 2.0 | 728.605 | 1.180 | 1.012 | 85 |
| 5 | 2.396909 | 2.0 | 745.453 | 1.211 | 1.003 | 418 |
| 6 | 0.599777 | 2.0 | 749.562 | 1.219 | 1.001 | 644 |
| 7 | 0.150042 | 2.0 | 750.587 | 1.221 | 1.000 | 699 |
| 8 | 0.037513 | 2.0 | 750.843 | 1.222 | 1.000 | 706 |

In the second test, the function $u$ to be projected is defined on the unit cube, by

$$u(x, y) = 2^6 x(1 - x)y(1 - y)z(1 - z) \quad (x, y, z) \in \Omega = [0, 1]^3.$$  

(4.5)
The weights are defined by

\[
\omega = \begin{cases} 
1 & \text{if } (x,y,z) \in [0.25,0.5] \cup [0.5,0.75], \\
\epsilon & \text{elsewhere on } \Omega.
\end{cases}
\]  

(4.6)

The convergence and stability of the weighted \(L^2\) projection is seen in Table 2.

**Table 2:** The weighted \(L^2\) projection for 3D function (4.5).

| \(|u - Q_h^\omega u|_{L^2}\|_h^n| \ |\|Q_h^\omega u||_{H^1} \| \ |C_1(4.3)\ | \ |C_2(4.4)\ | \ |\# CG\ |
|---|---|---|---|---|
| \(\epsilon = 10^{-6}\) in (4.6) | | | | |
| 4 | 0.004046 | 1.9 | 0.294 | 1.101 | 1.032 | 12 |
| 5 | 0.001072 | 1.9 | 0.298 | 1.077 | 1.009 | 19 |
| 6 | 0.000433 | 1.3 | 0.299 | 0.672 | 1.002 | 20 |

| \(\epsilon = 1\) in (4.6) | | | | |
| 4 | 0.016417 | 1.7 | 2.206 | 2.014 | 1.042 | 11 |
| 5 | 0.004333 | 1.9 | 2.152 | 1.919 | 1.011 | 9 |
| 6 | 0.001106 | 2.0 | 2.138 | 1.883 | 1.003 | 6 |

| \(\epsilon = 10^6\) in (4.6) | | | | |
| 4 | 15.917265 | 1.7 | 2186.042 | 2.058 | 1.043 | 41 |
| 5 | 4.211933 | 1.9 | 2130.689 | 1.955 | 1.011 | 169 |
| 6 | 1.076113 | 2.0 | 2116.865 | 1.916 | 1.003 | 255 |

Our third numerical test is on a function which violates the Poincaré inequality. This test function is similar to a counter example given in [8]:

\[
u(x,y) = \begin{cases} 
1 & \sqrt{x^2 + y^2} < c_1, \\
\xi(r) & c_1 \leq r = \sqrt{x^2 + y^2} \leq c_2, \\
0 & c_1 < \sqrt{x^2 + y^2},
\end{cases}
\]

(4.7)

where \(\xi(r)\) is polynomial of degree 10 satisfying

\[
\xi(c_1) = 1, \quad \xi^{(i)}(c_1) = \xi(c_2) = \xi^{(i)}(c_2) = 0 \quad \text{for } i = 1, 2, 3, 4.
\]

Two such functions are shown in Figure 9.

For simplicity and using the polar coordinates, we let \(\Omega = [-1,1]^2\) in the third test. We select the weight of \(L^2\) projection as, shown in Figure 10

\[
\omega = \begin{cases} 
\epsilon & \text{if } \max\{|x|,|y|\} < 1/2, \\
1 & \text{elsewhere}.
\end{cases}
\]

(4.8)

Here we have two \(\epsilon\):

\[
\epsilon = 10^{-2}, \ 10^{-8}.
\]
For these two $\epsilon$, the weighted norms are
\[
\|u\|_{L^2(\Omega)} \approx \|Q^\omega_h u\|_{L^2(\Omega)} = 1.0129, \quad |u|_{H^1(\Omega)} \approx 0.63011531, \quad \text{when } \epsilon = 10^{-2},
\]
\[
\|u\|_{L^2(\Omega)} \approx \|Q^\omega_h u\|_{L^2(\Omega)} = 1.0000000129, \quad |u|_{H^1(\Omega)} \approx 0.00063011531, \quad \text{when } \epsilon = 10^{-8},
\]
where $u$ is defined in (4.7) with $c_1 = \frac{3}{4}, \quad c_2 = 1$.

One can see the reasons why the weighted norms behave so in (4.9) and (4.10). Thus, it is clear why the Poincaré inequality fails:
\[
\|u\|_{L^2(\Omega)} \not\leq C |u|_{H^1(\Omega)}, \quad \text{and of course, } \|Q^\omega_h u\|_{L^2(\Omega)} \not\leq C |u|_{H^1(\Omega)}.
\]

Though the function $u$ in (4.7) fails to satisfy Poincaré inequality, its weighted $L^2$ projection can still approximate the function at the optimal order. In particular, (1.6) holds. This is clear in the Table 3 of numerical data. The proportional relation between the norms and $\epsilon$ can be seen in Table 3.

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Table 3: The weighted $L^2$ projection for 2D function (4.7).

| $\|u - Q_h^* u\|_{L^2}$ | $h^n$ | $|Q_h^* u|_{H^1}$ | $C_1(4.3)$ | $C_2(4.4)$ | # CG |
|--------------------------|------|------------------|------------|------------|------|
|                          |      |                  |            |            |      |
| $\epsilon = 10^{-2}$ in (4.8) |      |                  |            |            |      |
| 5                        | 0.006587 | 0.0          | 0.624      | 0.323      | 1.146 | 145 |
| 6                        | 0.002957 | 1.2          | 0.665      | 0.201      | 1.092 | 189 |
| 7                        | 0.000911 | 1.7          | 0.642      | 0.168      | 1.026 | 209 |
| 8                        | 0.000238 | 1.9          | 0.634      | 0.162      | 1.006 | 209 |
| $\epsilon = 10^{-8}$ in (4.8) |      |                  |            |            |      |
| 5                        | 0.00000658 | 0.0        | 0.000624   | 0.323      | 1.146 | 332 |
| 6                        | 0.00000296 | 1.2        | 0.000665   | 0.201      | 1.092 | 433 |
| 7                        | 0.00000091 | 1.7        | 0.000642   | 0.168      | 1.026 | 453 |
| 8                        | 0.00000024 | 1.9        | 0.000634   | 0.162      | 1.006 | 423 |

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References


