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# A New Approach for Solving Stokes Systems Arising From a Distributive Relaxation Method

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# A NEW APPROACH FOR SOLVING STOKES SYSTEMS ARISING FROM A DISTRIBUTIVE RELAXATION METHOD

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Abstract. The distributed relaxation method for the incompressible Stokes problem has been advertised as an adequate change of variables that leads to a lower triangular system with Laplace operators on the main diagonal for which multigrid methods are very efficient. We show that under high regularity of the Laplacian the transformed system admits almost block–lower triangular form. We analyze the distributed relaxation method and compare it with other iterative methods for solving the Stokes system. We also present numerical experiments illustrating the effectiveness of the transformation which is well–established for certain finite difference discretizations of Stokes problems.

## 1. Description of the problem

We consider the stationary Stokes equations:

$$\begin{array}{lll} -\Delta \mathbf{u} & +\nabla p & = \mathbf{f}, \text{ in } \Omega, \\ \operatorname{div} \mathbf{u} & & = g, \text{ in } \Omega, \end{array} \tag{1.1}$$

with vanishing Dirichlet boundary condition $\mathbf{u} = 0$ on $\partial\Omega$ and $g$ satisfying the constrain:

$$\int_\Omega g \, dx = 0.$$

The "distributive relaxation" as proposed in [2] (see also [7]), reads as follows. Make the following change of variables,

$$\begin{array}{ll} \mathbf{u} & = \mathbf{w} - \nabla \psi, \\ p & = -\Delta \psi. \end{array} \tag{1.2}$$

Substituting (1.2) into (1.1) (assuming enough regularity), leads to

$$\begin{array}{lll} -\Delta \mathbf{w} & & = \mathbf{f}, \\ \operatorname{div} \mathbf{w} & -\Delta \psi & = g. \end{array} \tag{1.3}$$

The transformed operator has a lower triangular form with Laplace operators on the main diagonal, for which efficient multigrid methods are available. Of course,

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we did not pay any attention to the boundary conditions. It is easily seen that if the original variable $\mathbf{u}$ vanishes on the boundary $\Gamma$ of the given domain $\Omega$, then the new variable $\mathbf{w}$ couples with $\nabla\psi$ on $\Gamma$. That is, the above system (1.3) does not actually decouple $\mathbf{w}$ from $\psi$. Our goal in this paper is to consider the coupled problem (1.1)–(1.2) in a weak form with proper Sobolev spaces for $\mathbf{u}$, and $\psi$ together with appropriate boundary conditions and then to find an efficient finite element discretization for the coupled (more elliptic) problem.

The remainder of the paper is organized as follows: In Section 2 we introduce and analyze an Uzawa type algorithm for solving (1.1). We establish sharp rates of convergence in terms of the inf-sup condition constant at the continuous level. Even though the convergence results of Uzawa Algorithms at the continuous level are known (see, e.g., [1, 4]), the presented analysis is based on fixed point approach, and will be used for establishing the convergence of the distributive relaxation iteration.

In Section 3, we study the properties of the transformed problem. An analysis of the distributed relaxation method as an iterative process is given, using the results of Section 2.

Finally, in Section 4 we report on a number of numerical tests illustrating the performance of the new method based on the transformed system. Some early results regarding multigrid convergence with distributed (transforming) smoothers are found in [8].

## 2. An Uzawa type algorithm

Let $\mathbf{V} := (H_0^1(\Omega))^d$,where $d = 2$ or $d = 3$, and $P = L_0^2(\Omega)$. We will also denote by $A(\mathbf{u}, \mathbf{v})$ the bilinear form

$$\begin{aligned} A(\mathbf{u}, \mathbf{v}) := (\nabla \mathbf{u}, \nabla \mathbf{v}) &= \sum_{i=1}^{d} (\nabla \mathbf{u}_i, \nabla \mathbf{v}_i) \\ &= (\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v}) + (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v}),\ \ \mathbf{u}, \mathbf{v} \in \mathbf{V}. \end{aligned}$$

We denote the norm induced by $A$ with $|\cdot|_{\mathbf{V}}$. The norm on $P$ is the $L^2$ standard norm and is simply denoted by $\|\cdot\|$. A variational formulation of (1.1) is
Find $\mathbf{u} \in \mathbf{V}, p \in P$ such that

$$\begin{array}{rll} A(\mathbf{u}, \mathbf{v}) & -(p, \operatorname{div} \mathbf{v}) & = (\mathbf{f}, \mathbf{v}),\ \ \mathbf{v} \in \mathbf{V}. \\ (\operatorname{div} \mathbf{u}, q) & & = (g, q),\ \ q \in P. \end{array} \tag{2.1}$$

It is known that the following "inf-sup" condition holds:

$$\inf_{p \in P} \sup_{\mathbf{v} \in \mathbf{V}} \frac{(p, \operatorname{div} \mathbf{v})}{\|p\| \ |\mathbf{v}|_{\mathbf{V}}} = c_0 > 0. \tag{2.2}$$

Here, and throughout this paper, the "inf" and "sup" are taken over nonzero vectors. Due to the choice of the norm on $\mathbf{V}$, it is clear that we have $c_0 \leq 1$.

Next, we define $F : P \to P$ by

$$F(p) = p + g - \operatorname{div}\mathbf{u}, \text{ where } \mathbf{u} \text{ solves} \tag{2.3}$$
$$A(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}) + (p, \operatorname{div}\mathbf{v}), \ \forall \mathbf{v} \in \mathbf{V}.$$

Let $\widehat{p}_0 = F(0) = g - \operatorname{div}\mathbf{u}_0$, where $\mathbf{u}_0$ is the solution of the problem

$$A(\mathbf{u}_0,\mathbf{v}) = (\mathbf{f},\mathbf{v}), \ \forall \mathbf{v} \in \mathbf{V},$$

and let $T : P \to P$ be the linear part of the mapping $F$, i.e.,

$$Tp = p - \operatorname{div}\mathbf{u}, \text{ where} \tag{2.4}$$
$$A(\mathbf{u},\mathbf{v}) = (p, \operatorname{div}\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}.$$

We have $F(p) = Tp + \widehat{p}_0$. If $p$ is a fixed point of $F$ and $\mathbf{u}$ solves the second equation in (2.3), then $(\mathbf{u}, p)$ is the solution of (2.1). The solution $(\mathbf{u}, p)$ can be found by the functional iteration of $F$. More precisely, we are led to the following Uzawa type algorithm.

**Algorithm 2.1** (Uzawa algorithm)**.**
*Let $p_0$ be any given initial approximation to $p$, and for $n = 1, 2, \cdots$ construct $(\mathbf{u}_n, p_n)$ by*

$$A(\mathbf{u}_n,\mathbf{v}) = (\mathbf{f},\mathbf{v}) + (p_{n-1}, \operatorname{div}\mathbf{v}), \ \mathbf{v} \in \mathbf{V},$$
$$p_n = F(p_{n-1}) = p_{n-1} + g - \operatorname{div}\mathbf{u}_n.$$

**Remark 2.2.** *The following modification of Uzawa algorithm, referred to as the "iterated penalty method" (cf., e.g., [3]) is sometimes used in practice. For any parameter $\beta \geq 0$, consider the iteration*

$$\begin{aligned} A(\mathbf{u}_n,\ \mathbf{v}) + \beta\,(\operatorname{div}\mathbf{u}_n - g,\ \operatorname{div}\mathbf{v}) &= (\mathbf{f},\mathbf{v}) + (p_{n-1},\ \operatorname{div}\mathbf{v}),\ \forall \mathbf{v} \in \mathbf{V}, \\ p_n &= p_{n-1} + g - \operatorname{div}\mathbf{u}_n. \end{aligned}$$

*Its convergence can be analyzed similarly to the (unmodified) Uzawa algorithm (see below), with the change $A(\mathbf{u},\ \mathbf{v}) := A(\mathbf{u},\ \mathbf{v}) + \beta\,(\operatorname{div}\mathbf{u},\ \operatorname{div}\mathbf{v})$ and $(\mathbf{f},\ \mathbf{v}) := (\mathbf{f},\ \mathbf{v}) + \beta\,(g,\ \operatorname{div}\ \mathbf{v})$.*

Algorithm 2.1 converges provided $\|T\| < 1$.

**Theorem 2.3.** $\|T\| = 1 - c_0^2$.

*Proof.* Let $p, q \in P$, and define $\mathbf{u}, \mathbf{w} \in \mathbf{V}$ as the solutions of

$$A(\mathbf{u},\mathbf{v}) = (p, \operatorname{div}\mathbf{v}) \ \text{ and } \ A(\mathbf{w},\mathbf{v}) = (q, \operatorname{div}\mathbf{v}), \ \forall \mathbf{v} \in \mathbf{V},$$

respectively. First, we note that

$$(Tp, q) = (Tp, Tq) + (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{w}).$$

Consequently, $T : P \to P$ is a a bounded, symmetric, and nonnegative definite operator. By noticing that

$$(Tp, p) = \|p\|^2 - (p, \operatorname{div} \mathbf{u}) = \|p\|^2 - A(\mathbf{u}, \mathbf{u}),$$

we obtain

$$\|T\| = \sup_{p \in P} \frac{(Tp, p)}{\|p\|^2} = \sup_{p \in P} \left(1 - \frac{A(\mathbf{u}, \mathbf{u})}{\|p\|^2}\right) = 1 - \inf_{p \in P} \frac{A(\mathbf{u}, \mathbf{u})}{\|p\|^2}.$$

On the other hand, if $G$ is the functional $v \to (p, \operatorname{div} v)$, then $G \in \mathbf{V}'$ and from Lax-Milgram theorem we get

$$A(\mathbf{u}, \mathbf{u}) = \|G\|^2_{\mathbf{V}'} = \sup_{\mathbf{v} \in \mathbf{V}} \frac{(p, \operatorname{div} \mathbf{v})^2}{|\mathbf{v}|^2_{\mathbf{V}}}.$$

Consequently

$$\|T\| = 1 - \inf_{p \in P} \sup_{\mathbf{v} \in \mathbf{V}} \frac{(p, \operatorname{div} \mathbf{v})^2}{\|p\|^2 \ |\mathbf{v}|^2_{\mathbf{V}}} = 1 - c_0^2.$$

□

**Corollary 2.4.** *The sequence* $\{(\mathbf{u}_n, p_n)\}$ *given by the Algorithm* (2.1) *converges to the solution* $(\mathbf{u}, p)$ *of* (2.1) *and*

$$\|p_n - p\| \le (1 - c_0^2)\|p_{n-1} - p\|,$$

*and*

$$\|\mathbf{u}_n - \mathbf{u}\|_{\mathbf{V}} \le \|p_{n-1} - p\|.$$

Note that if the second equation in (2.1) is replaced by

$$p_n = p_{n-1} + \alpha(g - \operatorname{div} \mathbf{u}_n),$$

with $\alpha \in (0, 2)$, then we still get convergence of the new algorithm. The operator $T_\alpha$ which gives the convergence factor of the new algorithm is defined by

$$\begin{aligned} T_\alpha p &= p - \alpha \operatorname{div} \mathbf{u}, \text{ where} \\ A(\mathbf{u}, \mathbf{v}) &= (p, \operatorname{div} \mathbf{v}), \ \mathbf{v} \in \mathbf{V}. \end{aligned} \tag{2.5}$$

$T_\alpha$ remains a symmetric operator and as before we get

$$\begin{aligned} (T_\alpha p, p) &= \|p\|^2 - \alpha \ (p, \operatorname{div} \mathbf{u}) = \|p\|^2 - \alpha A(\mathbf{u}, \mathbf{u}) \\ &= \|p\|^2 - \alpha \sup_{\mathbf{v} \in \mathbf{V}} \frac{(p, \operatorname{div} \mathbf{v})^2}{|\mathbf{v}|^2_{\mathbf{V}}}. \end{aligned}$$

The following estimate

$$\sup_{p \in P} \sup_{\mathbf{v} \in \mathbf{V}} \frac{(p, \operatorname{div} \mathbf{v})^2}{\|p\|^2 |\mathbf{v}|^2_{\mathbf{V}}} = 1,$$

together with the in-sup condition,

$$\inf_{p \in P} \sup_{\mathbf{v} \in \mathbf{V}} \frac{(p, \operatorname{div} \mathbf{v})^2}{\|p\|^2 |\mathbf{v}|^2_{\mathbf{V}}} = c_0^2,$$

lead to

**Theorem 2.5.**

$$\|T_\alpha\| = 1 - \alpha\ c_0^2,\ \textit{for}\ \ 0 < \alpha \le 1,\ \textit{and}$$
$$\|T_\alpha\| = \max\{\alpha - 1, |1 - \alpha c_0^2|\},\ \textit{for}\ \ 1 < \alpha < 2.$$

*The optimal rate of convergence is obtained when* $\alpha = 2/(1 + c_0^2)$ *and in this case the convergence factor is*

$$\|T_{opt}\| = \frac{1 - c_0^2}{1 + c_0^2}.$$

**Remark 2.6.** *The Uzawa algorithms provides a solution to the fixed point problem,* $F(p) = p$. *In terms of* $T$, *it reads* $(I - T)p = \widehat{p}_0 = F(0)$. *Thus, we are inverting* $I - T$ *by a stationary iterative method, which converges, if* $\|T\| < 1$. *Since,* $I - T$ *is s.p.d., we can use the CG (conjugate gradient) method instead. Its convergence rate is bounded by* $\varrho_{T_{CG}} \le \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$ *where* $\kappa \le \frac{1}{1-\|T\|}$ *is an upper bound of the condition number of* $I - T$. *In the present case, we have*

$$\varrho_{T_{CG}} \le \frac{1 - c_0}{1 + c_0}.$$

## 3. Distributive relaxation method as an iterative process

For this section we assume that $\Omega$ is a convex polygonal domain. We return to the problem (1.1) with the change of variable (1.2). To be more precise, we consider the change of variable $(\mathbf{u}, p) \to (\mathbf{w}, \psi)$ where

$$\begin{aligned} -\Delta\psi &= p,\ \text{in } \Omega, \\ \frac{\partial\psi}{\partial n} &= 0 \text{ on } \Gamma, \end{aligned} \tag{3.1}$$

and

$$\mathbf{w} = \mathbf{u} + \nabla\psi. \tag{3.2}$$

Then, assuming enough regularity, (1.1) becomes

$$\begin{aligned} -\Delta\mathbf{w} &= \mathbf{f},\ \text{in } \Omega, \\ \mathbf{w} &= \nabla\psi,\ \text{on } \Gamma, \\ -\Delta\psi &= g - \operatorname{div}\mathbf{w},\ \text{in } \Omega, \\ \frac{\partial\psi}{\partial n} &= 0,\ \text{on } \Gamma. \end{aligned} \tag{3.3}$$

Let $H_*^1 = H_*^1(\Omega) = \{\varphi \in H^1 \mid (\varphi, 1) = 0\}$ and

$$H_*^2 = H_*^2(\Omega) = \left\{\varphi \in H^2 \mid (\varphi, 1) = 0, \left.\frac{\partial\varphi}{\partial n}\right|_\Gamma = 0\right\}.$$

On $H^2_*$ we consider the inner product given by

$$(\varphi, \psi)_2 := (\Delta\varphi, \Delta\psi).$$

The norm on $H^2_*$ is denoted by $\|\cdot\|_2$ and is the norm induced by the $(\cdot,\cdot)_2$-inner product. A variational formulation of (3.3) reads: Find $\mathbf{w} \in H^1(\Omega)$, $\psi \in H^2_*$, such that

$$\begin{aligned} (\nabla \mathbf{w}, \nabla \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \ \text{ for all } \ \mathbf{v} \in \mathbf{V}, \\ \mathbf{w} &= \nabla \psi, \ \text{ on } \Gamma, \end{aligned} \tag{3.4}$$

and

$$(\nabla \psi, \nabla q) = (g - \operatorname{div} \mathbf{w}, q) \ \text{ for all } q \in H^1_* \ . \tag{3.5}$$

The solution of problem (3.4)-(3.5) can be also written in terms of a fixed point problem on $H^2_*$. Define $F : H^2_* \to H^2_*$ by $F(\varphi) = \psi$ where we first compute $\mathbf{w}$ by solving

$$\begin{aligned} (\nabla \mathbf{w}, \nabla \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \ \text{ for all } \ \mathbf{v} \in \mathbf{V}, \\ \mathbf{w} &= \nabla \varphi \text{ on } \Gamma, \end{aligned} \tag{3.6}$$

and then $\psi$ is computed from

$$(\nabla \psi, \nabla q) = (g - \operatorname{div} \mathbf{w}, q) \ \text{ for all } q \in H^1_* \ . \tag{3.7}$$

If $\widehat{\psi}_0 = F(0)$, then $F(\varphi) = S\varphi + \widehat{\psi}_0$ where $S : H^2_* \to H^2_*$ is defined as the linear part of $F$, i.e., $S(\varphi) = \psi$, where

$$\begin{aligned} (\nabla \mathbf{w}, \nabla \mathbf{v}) &= 0 \ \text{ for all } \ \mathbf{v} \in \mathbf{V}, \\ \mathbf{w} &= \nabla \varphi \quad \text{ on } \Gamma, \end{aligned} \tag{3.8}$$

and $\psi$ is given by

$$(\nabla \psi, \nabla q) = -(\operatorname{div} \mathbf{w}, q) \ \text{ for all } q \in H^1_* \ . \tag{3.9}$$

We note that if $\psi$ is a fixed point of $F$ then $(\mathbf{w}, \psi)$, with $\mathbf{w}$ defined by be the solution of (3.6) with $\varphi = \psi$, is the solution of (3.4)-(3.5). The existence and the uniqueness of the solution of (3.4)-(3.5) follows from the existence and the uniqueness of the solution of the Stokes system.

**Algorithm 3.1** (Distributed relaxation)**.**
*Let $\psi_0$ be any given initial approximation to $\psi$. For $n = 1, 2, \cdots$, we construct $(\mathbf{w}_n, \psi_n)$ by first solving for $\mathbf{w}_n$*

$$\begin{aligned} (\nabla \mathbf{w}_n, \nabla \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \ \textit{ for all } \ \mathbf{v} \in \mathbf{V}, \\ \mathbf{w}_n &= \nabla \psi_{n-1}, \quad \textit{on } \Gamma, \end{aligned} \tag{3.10}$$

*and then $\psi_n$ is computed from*

$$(\nabla \psi_n, \nabla q) = (g - \operatorname{div} \mathbf{w}_n, q) \ \textit{ for all } q \in H^1_* \ . \tag{3.11}$$

The above algorithm converges provided $\|S\| < 1$.

**Theorem 3.2.** $\|S\|_2 = 1 - c_0^2$.

*Proof.* First we observe that $S$ is a bounded operator from $H_*^2$ to $H_*^2$. Next, we prove that for $\varphi_1, \varphi_2 \in H_*^2 \cap H^3$ we have

$$(S\varphi_1, S\varphi_2)_2 = (\varphi_1, S\varphi_2)_2 - (\operatorname{curl} \mathbf{w}_1, \operatorname{curl} \mathbf{w}_2), \tag{3.12}$$

where $\mathbf{w}_i$ and $\psi_i = S\varphi_i$, $i = 1, 2$, are defined

$$(\nabla \mathbf{w}_i, \nabla \mathbf{v}) = 0 \quad \text{for all} \quad \mathbf{v} \in \mathbf{V},$$
$$\mathbf{w}_i = \nabla \varphi_i \text{ on } \Gamma,$$

and

$$(\nabla \psi_i, \nabla q) = -(\operatorname{div} \mathbf{w}_i, q) \quad \text{for all } q \in H_*^1 .$$

The main ingredient in proving (3.12) is the fact that for sufficiently smooth functions $\mathbf{w}$ with $\Delta \mathbf{w} = 0$ we have

$$\operatorname{curl} \operatorname{curl} \mathbf{w} = \nabla(\operatorname{div} \mathbf{w}). \tag{3.13}$$

For the 2D case for example, the proof of (3.12) is seen as follows:

$$\begin{aligned}
(S\varphi_1, S\varphi_2)_2 &= (\psi_1, \psi_2)_2 = (\Delta\psi_1, \Delta\psi_2) = (\operatorname{div} \mathbf{w}_1, \operatorname{div} \mathbf{w}_2) \\
&= -(\mathbf{w}_1, \nabla(\operatorname{div} \mathbf{w}_2)) = -(\mathbf{w}_1, \operatorname{curl} \operatorname{curl} \mathbf{w}_2) \\
&= -(\operatorname{curl} \mathbf{w}_1, \operatorname{curl} \mathbf{w}_2) + (\mathbf{w}_1 \cdot \tau, \operatorname{curl} \mathbf{w}_2)_\Gamma \\
&= -(\operatorname{curl} \mathbf{w}_1, \operatorname{curl} \mathbf{w}_2) + (\nabla\varphi_1 \cdot \tau, \operatorname{curl} \mathbf{w}_2)_\Gamma \\
&= -(\operatorname{curl} \mathbf{w}_1, \operatorname{curl} \mathbf{w}_2) - (\nabla\varphi_1, \operatorname{curl} \operatorname{curl} \mathbf{w}_2) \\
&= -(\operatorname{curl} \mathbf{w}_1, \operatorname{curl} \mathbf{w}_2) - (\nabla\varphi_1, \nabla(\operatorname{div} \mathbf{w}_2)) \\
&= -(\operatorname{curl} \mathbf{w}_1, \operatorname{curl} \mathbf{w}_2) + (\Delta\varphi_1, \operatorname{div} \mathbf{w}_2) \\
&= -(\operatorname{curl} \mathbf{w}_1, \operatorname{curl} \mathbf{w}_2) + (\Delta\varphi_1, \Delta\psi_2) \\
&= -(\operatorname{curl} \mathbf{w}_1, \operatorname{curl} \mathbf{w}_2) + (\varphi_1, S\varphi_2)_2.
\end{aligned}$$

Here, $\tau$ represents the tangential vector to $\Gamma$. For the 3D case, " $\cdot\, \tau$" is replaced by " $\times\, \tau$". From (3.12) and the density of $H_*^2 \cap H^3$ in $H_*^2$, we conclude that $S$ is a symmetric operator and in particular,

$$(S\varphi, \varphi)_2 = (S\varphi, S\varphi)_2 + (\operatorname{curl} \mathbf{w}, \operatorname{curl} \mathbf{w}),$$

which proves that $S$ is a nonnegative operator. Thus,

$$\|S\|_2 = \sup_{\varphi \in H_*^2} \frac{(S\varphi, \varphi)_2}{\|\varphi\|_2^2} = \sup_{\varphi \in H_*^2} \frac{(\operatorname{div} \mathbf{w}, \Delta\varphi)}{\|\Delta\varphi\|^2},$$

where $\mathbf{w}$ is defined by (3.8). Assuming that $\varphi \in H_*^2 \cap H^3$, we let $\mathbf{u} = \mathbf{w} - \nabla\varphi$. It follows that

$$\operatorname{div} \mathbf{w} = \operatorname{div} \mathbf{u} + \Delta\varphi,$$

and

$$-\Delta \mathbf{u} = \nabla(\Delta\varphi).$$

The variational form of the above identity is

$$A(\mathbf{u}, \mathbf{v}) = (-\Delta\varphi, \operatorname{div} \mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{V}.$$

In particular, taking $\mathbf{v} = \mathbf{u}$, we get

$$A(\mathbf{u}, \mathbf{u}) = (-\Delta\varphi, \operatorname{div} \mathbf{u}).$$

It follows that:

$$\begin{aligned} \|S\|_2 &= \sup_{\varphi \in H^2_* \cap H^3} \frac{(S\varphi, \varphi)_2}{\|\varphi\|_2^2} = \sup_{\varphi \in H^2_* \cap H^3} \frac{(\operatorname{div} \mathbf{u} + \Delta\varphi, \Delta\varphi)}{\|\Delta\varphi\|^2} \\ &= \sup_{\varphi \in H^2_*} \left(1 - \frac{A(\mathbf{u}, \mathbf{u})}{\|\Delta\varphi\|^2}\right). \end{aligned}$$

Denoting $-\Delta\varphi = q$, we have $q \in L^2_0 = P$ and

$$A(\mathbf{u}, \mathbf{v}) = (q, \operatorname{div} \mathbf{v}), \;\; \mathbf{v} \in \mathbf{V}.$$

Therefore (see the proof of Theorem 2.3),

$$\|S\|_2 = \sup_{q \in P} \left(1 - \frac{A(\mathbf{u}, \mathbf{u})}{\|q\|^2}\right) = 1 - \inf_{q \in P} \frac{A(\mathbf{u}, \mathbf{u})}{\|q\|^2} = 1 - c_0^2.$$

□

This result proves that the distributive relaxation method as an iterative method has similar convergence properties with the Uzawa method. We can also modify the Algorithm (3.14)- (3.15) to obtain a Distributive Relaxation algorithm with a faster rate of convergence. The new algorithm reads as follows: Let $\alpha \in (0, 2)$ and let $\psi_0$ be any approximation of $\psi$. For $n = 1, 2, \cdots$ construct $(\mathbf{w}_n, \psi_n)$ by

$$\begin{aligned} (\nabla \mathbf{w}_n, \nabla \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \;\; \text{for all } \;\mathbf{v} \in \mathbf{V}, \\ \mathbf{w}_n &= \alpha \, \nabla\psi_{n-1}, \quad \text{on } \Gamma, \end{aligned} \tag{3.14}$$

and $\psi_n$

$$\alpha \, (\nabla\psi_n, \nabla q) = (g - \operatorname{div} \mathbf{w}_n, q) \;\; \text{for all } q \in H^1_* \,. \tag{3.15}$$

The operator $S_\alpha$ involved in the convergence analysis is $S_\alpha : H^2_* \to H^2_*$ defined by $S_\alpha\varphi = \psi$ where

$$\begin{aligned} (\nabla \mathbf{w}, \nabla \mathbf{v}) &= 0 \;\; \text{for all } \;\mathbf{v} \in \mathbf{V}, \\ \mathbf{w} &= \alpha \, \nabla\varphi, \quad \text{on } \Gamma, \end{aligned} \tag{3.16}$$

and

$$\alpha \, (\nabla\psi, \nabla q) = -(\operatorname{div} \mathbf{w}, q) \;\; \text{for all } q \in H^1_* \,. \tag{3.17}$$

Following the proof of Theorem 3.2 we can see that $S_\alpha$ satisfies

$$(S_\alpha\varphi, \varphi)_2 = \|\Delta\varphi\|^2 - \alpha^{-2} A(\mathbf{u}, \mathbf{u}).$$

Thus, a similar version of Theorem 2.5 holds:

**Theorem 3.3.** $\|S_\alpha\|_2 < 1$ *for* $\alpha \in (0, 2)$*. The optimal* $\alpha$ *is*

$$\alpha = \sqrt{\frac{1+c_0^2}{2}},$$

*and the optimal convergence factor for the modified distributive relaxation algorithm is*

$$\|S_\alpha\|_2 = \frac{1-c_0^2}{1+c_0^2}.$$

**Remark 3.4.** *Similarly to Remark 2.6, we can use CG method instead of stationary iteration process to solve the fixed point problem* $F(\varphi) = \varphi$*, which in terms of* $S$ *reads* $(I - S)\varphi = F(0) = \widehat{\psi}_0$*. The operator* $I - S$ *is s.p.d. in the inner product* $(\cdot,\ \cdot)_2 = (\Delta\cdot,\ \Delta\cdot)$*. The use of CG will require applying Laplace operator to compute the inner products involved in the CG method. With this comment in mind, the same converge rate as in Remark 2.6 holds here as well.*

## 4. Numerical experiments

In this section, we present numerical tests for our new algorithm. We present the test in two parts. We first apply the traditional $P_4$-$P_3$ mixed finite element for the Stokes equations, where the velocity is approximated by $C^0$-$P_4$ polynomials and the pressure by $C^{-1}$-$P_3$ polynomials. Such a finite element pair satisfies the uniform (in grid size $h$) inf-sup condition, and produces pointwise divergence-free solutions for the velocity. The linear system of finite element equations is solved by the iterated penalty method (mentioned in Remark 2.2, see also below), with an inner multigrid iteration. In the second part of our test, we derive the same $P_4$-$P_3$ mixed finite element solutions by the distributive relaxation method (i.e., using the weak variational formulation (3.4)–(3.5)). More specifically, we represent the finite element solutions $(\mathbf{u}_h, p_h)$ of the original problem with the $(C^0$-$P_4$, $C^1$-$P_5)$ solutions $(\mathbf{w}_h, \psi_h)$ of the weak formulation of the transformed problem. Here $C^1$-$P_5$ stands for the well known Argyris element of continuously differentiable, piecewise $P_5$ polynomials.

We solve numerically the stationary Stokes equations (1.1). In the numerical test, the domain $\Omega$ is a unit square $(0, 1)^2$. We choose the exact solution (see Figure 4.1 and Figure 4.2):

$$\mathbf{u} = \operatorname{curl} s(x, y) = \begin{pmatrix} s_y \\ -s_x \end{pmatrix}, \quad p = -s_{xx}(x, y), \tag{4.1}$$

where

$$s(x, y) = 2^8(x - x^2)^2(y - y^2)^2.$$

Therefore $\int_\Omega p d\mathbf{x} = -\int_\Omega s_{xx} d\mathbf{x} = 0$ and in (1.1), $g = 0$ and

$$\mathbf{f} = \begin{pmatrix} -s_{yxx} - s_{yyy} - s_{xxx} \\ s_{xxx} + s_{xyy} - s_{xxy} \end{pmatrix}.$$

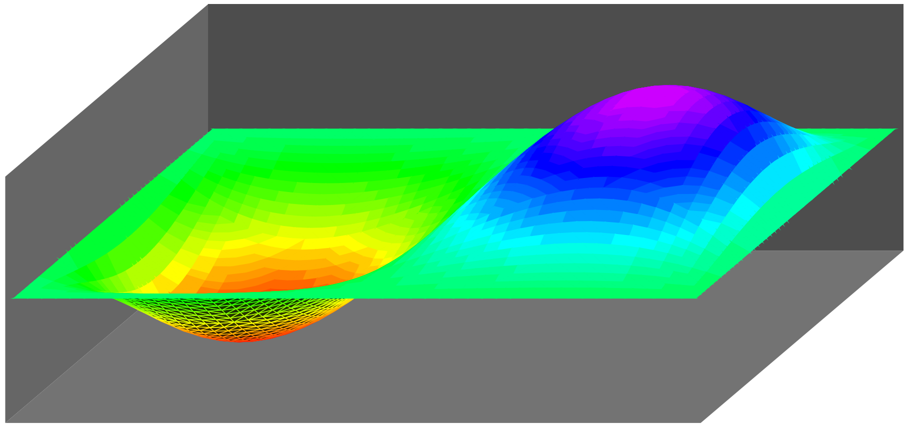

FIGURE 4.1. The first component of $\mathbf{u}$ in numerical test (4.1).

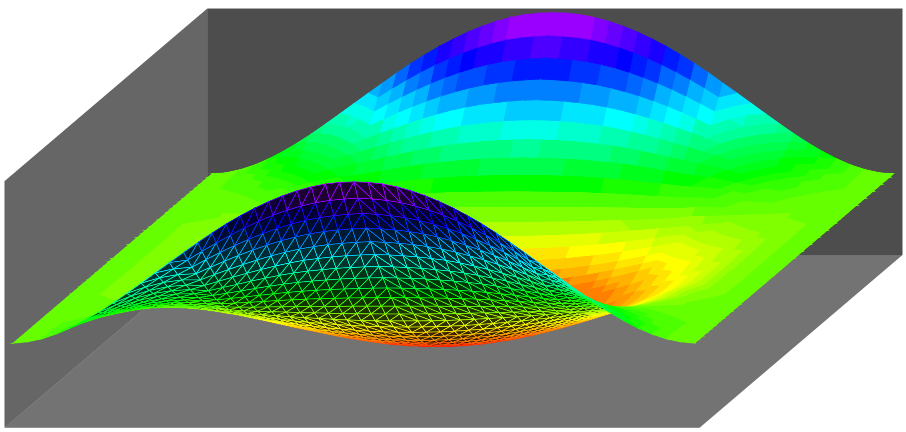

FIGURE 4.2. The solution $p$ in numerical test (4.1).

We apply several steps of successive refinement to an initial triangulation of $\Omega$, shown in Figure 4.3. We denote the grids by $\{T_h\}$. We use the standard $P_4$-$P_3$ mixed element, cf. [5], in the computation, i.e., the finite element spaces approximating the velocity and the pressure are, respectively,

$$\mathbf{V}_h = \left\{\mathbf{v}_h \in (H_0^1(\Omega))^2 \mid \mathbf{v}|_T \in (P_4)^2 \ \forall T \in T_h\right\} \subset \mathbf{V}, \tag{4.2}$$

$$P_h = \left\{q_h \in L_0^2(\Omega) \mid q_h = \operatorname{div} \mathbf{v}_h \ \text{ for } \ \mathbf{v}_h \in \mathbf{V}_h\right\} \subset P. \tag{4.3}$$

We note that, as proved by Scott and Vogelius [5], when the underlying grid $T_h$ is free of singular vertex, $P_h$ is precisely the full space of discontinuous piecewise $P_3$ polynomials with zero mean value. However, for our grids shown in Figure 4.3, we do have two singular vertices, $(0,0)$ and $(1,1)$, where a $P_h$ function has value

0. Alternatively, for the grids in Figure 4.3, $P_h$ is also defined by

$$P_h = \left\{ q_h \in L_0^2(\Omega) \mid q_h|_T \in P_3 \ \forall T \in T_h, \text{ and } \ q_h(0,0) = q_h(1,1) = 0 \right\}.$$

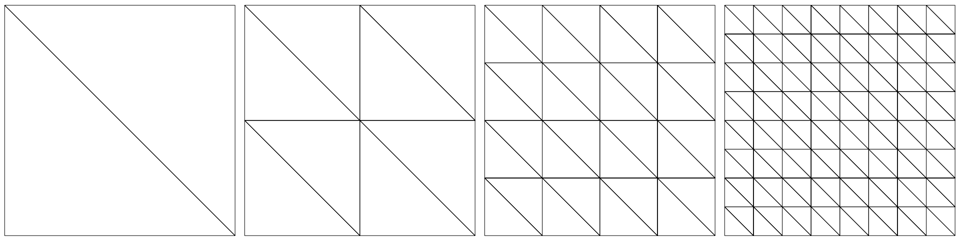

FIGURE 4.3. The multilevel grids in numerical computation.

The finite element approximation problems for (1.1) read: Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in P_h$ such that

$$A(\mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{4.4}$$

$$(q_h, \operatorname{div} \mathbf{u}_h) = 0 \qquad \forall q_h \in P_h. \tag{4.5}$$

The system of linear equations (4.4)–(4.5) are solved by the following iterated penalty method.

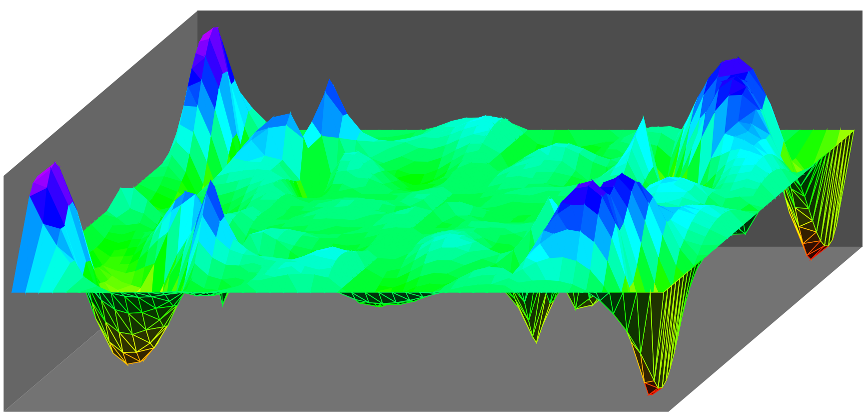

FIGURE 4.4. The velocity error on level 3 (Figure 4.3), $(\mathbf{u} - \mathbf{u}_h)_1$, by (4.6).

The iterated penalty method (mentioned in Remark 2.2) in the present setting, takes the following particular form.

**Definition 4.1.** (The iterated penalty method.) Let the initial iterate be $\mathbf{u}_h^0 = \mathbf{0}$. The rest iterates $\mathbf{u}_h^n$ are defined sequentially to be the unique solution of

$$a(\mathbf{u}_h^n, \mathbf{v}_h) + \beta(\operatorname{div} \mathbf{u}_h^n, \operatorname{div} \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) - (\operatorname{div} \sum_{j=0}^{n-1} \beta \mathbf{u}_h^j, \operatorname{div} \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{4.6}$$

$n = 1, 2, \cdots$. Here $\beta$ is positive constant. At the end of iteration, we let

$$p_h^n = -\operatorname{div} \sum_{j=0}^{n} \beta \mathbf{u}_h^j.$$

$\square$

**Remark 4.1.** *In the iterated penalty method, we only implement the $C^0$-$P_4$ finite element for the vector Laplacian-like equations. The solution $p_h$ is then obtained as a byproduct.*

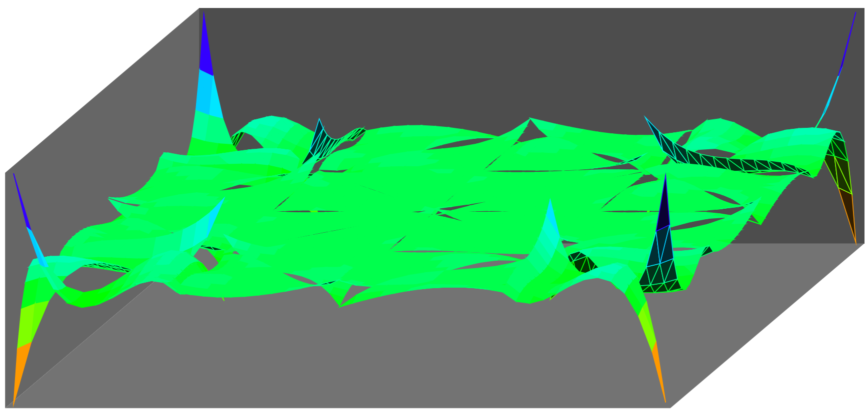

FIGURE 4.5. The pressure error on level 3 (Figure 4.3), $p - p_h$, by (4.6).

In our numerical test, we chose in (4.6)

$$\beta = 500, \text{ and } n = 4,$$

i.e., only 4 steps of iterated penalty method are applied. In addition, we apply the multigrid method to the Laplace-like equation (4.6). It is shown (e.g., Brenner and Scott [3], or by our analysis based on Remark 2.2)) that the iterated penalty method converges with a constant rate (depending on $\beta$ and the uniform inf-sup condition.) Combined with the optimal order multigrid solver, the finite element systems of equations (4.4) are solved in optimal order of arithmetic operations, i.e., in $O(N)$ steps where $N$ is the number of unknown values of $\mathbf{u}_h$. In Figure 4.4, the error $(\mathbf{u} - \mathbf{u}_h)_1$ for the first component of finite element solution $\mathbf{u}_h$ on the third level grid, shown in Figure 4.3, is depicted.

In Figure 4.5, the error for the pressure, $p - p_h$ on the third level grid, shown in Figure 4.3, is depicted. We note that big nodal error occurs at boundary, especially the four corners. It is common for discontinuous pressure finite elements to have bigger $L^\infty$ error, compared with continuous pressure finite elements. But the former would provide better solutions in the case of less regularity. In addition, we also plot the third level solution $p_h$ in Figure 4.6.

In Table 1, we list the errors and the the orders of convergence for the finite element solutions to (4.4), obtained by the iterated penalty method (4.6). All the

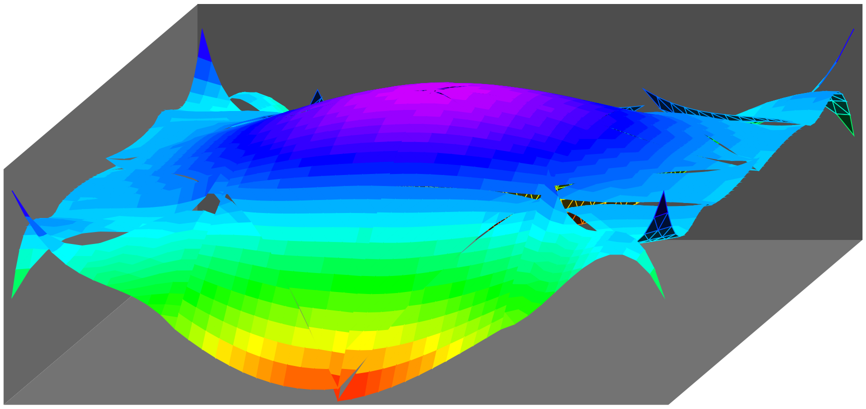

FIGURE 4.6. The level 3 solution $p_h$ by (4.6).

TABLE 1. The errors and orders of convergence, by (4.4) and (4.6).

| level | $\|\mathbf{u}-\mathbf{u}_h\|_{L^\infty}$ | $h^m$ | $\|\mathbf{u}-\mathbf{u}_h\|_{L^2}$ | $h^m$ | $\|\mathbf{u}-\mathbf{u}_h\|_{(H^1)^2}$ | $h^m$ |
|---|---|---|---|---|---|---|
| 1 | 3.10712 | | 2.16090 | | 16.85397 | |
| 2 | 0.19355 | 4.00 | 0.14208 | 3.93 | 2.39238 | 2.82 |
| 3 | 0.01583 | 3.61 | 0.00554 | 4.68 | 0.21127 | 3.50 |
| 4 | 0.00070 | 4.50 | 0.00015 | 5.20 | 0.01196 | 4.14 |
| | $\|p-p_h\|_{L^\infty}$ | $h^m$ | $\|p-p_h\|_{L^2}$ | $h^m$ | | |
| 1 | 341.23126 | | 94.63412 | | | |
| 2 | 38.09799 | 3.16 | 8.32775 | 3.51 | | |
| 3 | 5.86842 | 2.70 | 0.66941 | 3.64 | | |
| 4 | 0.52093 | 3.49 | 0.03419 | 4.29 | | |

orders of convergence match those provided by the theory, cf. [5]. They are all optimal, i.e., $O(h^4)$ for the $\mathbf{u}_h$ error in $H^1$, and $O(h^5)$ in $L^2$ and $L^\infty$, and for the pressure $p_h$ error, $O(h^4)$ in $L^2$ and $L^\infty$ norms.

In the second part of our numerical test, we use the distributive relaxation method defined in (3.1) – (3.3) to solve the test problem (1.1) with (4.1). In addition to the $C^0$-$P_4$ space $\mathbf{V}_h$ and the $C^{-1}$-$P_3$ space $P_h$ defined in (4.2) and (4.3), we need the following Argyris $C^1$ finite element space:

$$S_h = \left\{ \psi_h \in H^2(\Omega) \mid \psi_h|_T \in P_5, \ \forall T \in T_h, \ \text{and} \ \int_\Omega \psi_h d\mathbf{x} = 0 \right\}. \tag{4.7}$$

In the distributive relaxation method, we solve the following ($C^0$-$P_4$, $C^1$-$P_5$) coupling problems: Find $(\mathbf{w}_h, \psi_h)$ such that $(\mathbf{w}_h - \nabla\psi_h) \in \mathbf{V}_h$, $\psi_h \in S_h$, and that

$$(\nabla(\mathbf{w}_h - \nabla\psi_h), \nabla \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{4.8}$$

$$(\nabla\psi_h, \nabla\phi_h) = (-\operatorname{div}\mathbf{w}_h, \phi_h) \quad \forall \phi_h \in S_h. \tag{4.9}$$

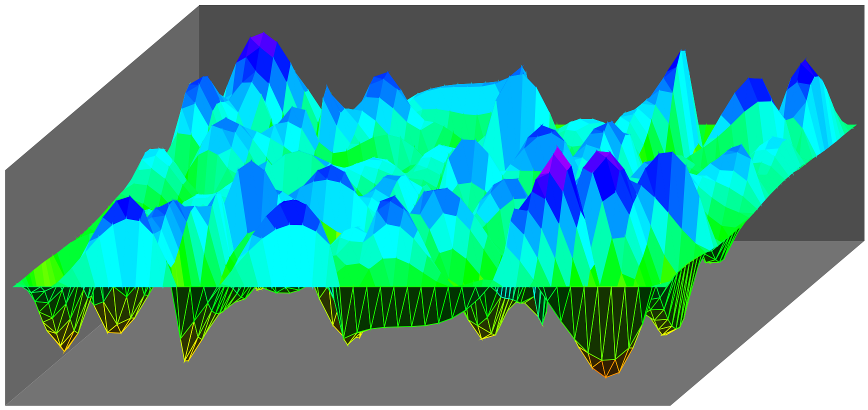

FIGURE 4.7. The level 3 error $(\mathbf{u} - \mathbf{u}_h)_1$, by (4.8)–(4.10).

The finite element solutions for the original Stokes equations are defined by

$$\mathbf{u}_h = \mathbf{w}_h - \nabla\psi_h \in \mathbf{V}_h, \quad \text{and} \quad p_h = \operatorname{div}\mathbf{w}_h \in P_h. \tag{4.10}$$

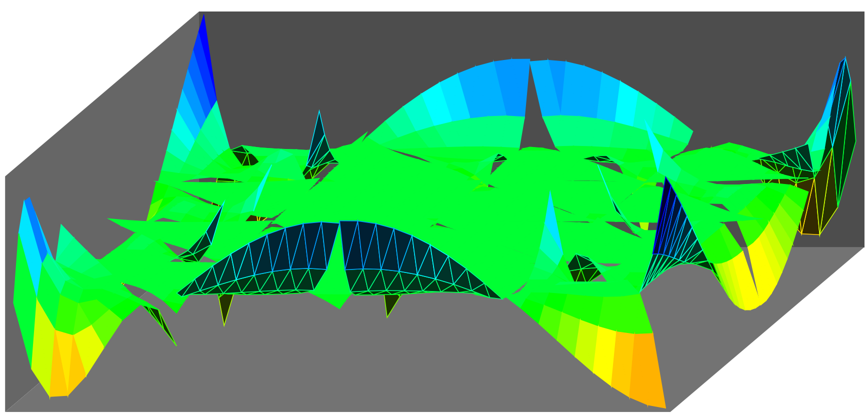

FIGURE 4.8. The level 3 error $p - p_h$, by (4.8)–(4.10).

When solving the coupled system (4.8)–(4.9), we apply the iterative algorithm in (3.10)–(3.11). The algorithm is shown to have a constant rate of convergence, depending only on the inf-sup constant. That is, in our numerical test, we do 10 steps of following Gauss-Seidel iterations, with $\psi_h^n = 0$: for $n = 1, 2, ..., 10$,

$$(\nabla(\mathbf{w}_h^n - \nabla\psi_h^{n-1}), \nabla\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{4.11}$$

$$(\nabla\psi_h^n, \nabla\phi_h) = (-\operatorname{div}\mathbf{w}_h^n, \phi_h) \quad \forall \phi_h \in S_h. \tag{4.12}$$

In Figure 4.7, we plot the error of the first component of $\mathbf{u}_h$ on the third level, obtained by (4.8)–(4.10). Comparing the two errors in Figure 4.4 and Figure 4.7), by the two methods, see also Table 2, we can see that the new method has much smaller nodal errors for $\mathbf{u}_h$ at the boundary of the domain $\Omega$. The same phenomenon happens to the pressure solution $p_h$. In Figure 4.8), the third level

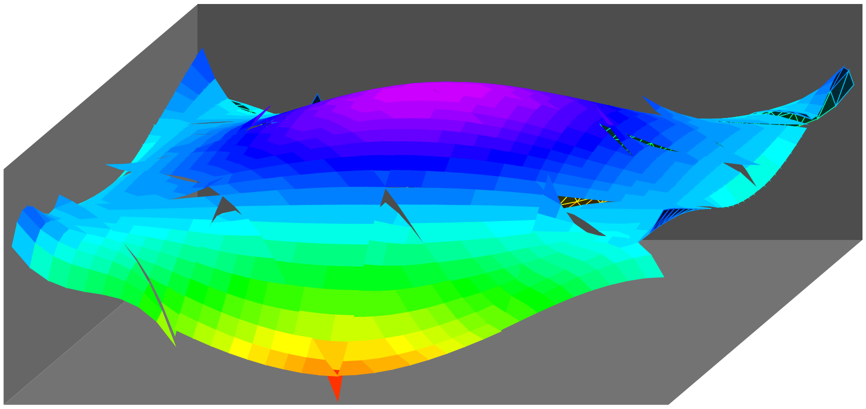

FIGURE 4.9. The level 3 solution $p_h$, by (4.8)–(4.10).

error for the pressure $p_h$ from (4.8)–(4.10) is depicted. Comparing Figures 4.5) and 4.8), the nodal error for the new method (4.8)–(4.9) is also smaller than that obtained by (4.4)–(4.5). In Figure 4.9), the third level solution for $p_h$ in (4.8)–(4.10) is depicted. We may compare it with that in Figure 4.6).

In Table 2, we list the errors and the the orders of convergence for the finite element solutions to (4.4), obtained by (4.10) with the iteration (4.11)–(4.12). All the orders of convergence are optimal, i.e., $O(h^4)$ for the $\mathbf{u}_h$ error in $H^1$, and $O(h^5)$ in $L^2$ and $L^\infty$, and for the pressure $p_h$ error, $O(h^4)$ in $L^2$ and $L^\infty$ norms.t-2, Comparing to the data in Table 1, the new method (4.8)–(4.9) is slightly better than the original mixed finite element method (4.4)–(4.5).

TABLE 2. The errors and orders of convergence, by (4.10) and (4.11)–(4.12).

| level | $\lvert\mathbf{u}-\mathbf{u}_h\rvert_{L^\infty}$ | $h^m$ | $\lvert\mathbf{u}-\mathbf{u}_h\rvert_{L^2}$ | $h^m$ | $\lvert\mathbf{u}-\mathbf{u}_h\rvert_{(H^1)^2}$ | $h^m$ |
|---|---|---|---|---|---|---|
| 1 | 0.85606886 | | 0.43641566 | | 3.63567362 | |
| 2 | 0.08656817 | 3.31 | 0.03997762 | 3.45 | 0.79585096 | 2.19 |
| 3 | 0.00544815 | 3.99 | 0.00179199 | 4.48 | 0.08349327 | 3.25 |
| 4 | 0.00028759 | 4.24 | 0.00009114 | 4.30 | 0.00682338 | 3.61 |
| | $\lVert p-p_h\rVert_{L^\infty}$ | $h^m$ | $\lVert p-p_h\rVert_{L^2}$ | $h^m$ | | |
| 1 | 25.32609624 | | 6.91154459 | | | |
| 2 | 8.02465563 | 1.66 | 1.14672539 | 2.59 | | |
| 3 | 1.34588347 | 2.58 | 0.09082632 | 3.66 | | |
| 4 | 0.11132024 | 3.60 | 0.00543629 | 4.06 | | |

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