A family of continuously differentiable finite elements on simplicial grids in four space dimensions

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Abstract

A family of continuously differentiable piecewise polynomials of degree k, for all $k \geq 17$, on general 4D simplicial grids, is constructed. Such a finite element space assumes full order of approximation. As a byproduct, we obtain a family of special 3D C_2-P_k elements on tetrahedral grids.

Keywords. continuously differentiable finite element, high order element, 4D biharmonic equation, simplicial grid.

AMS subject classifications (2000). 65N30, 73V05.

1 Introduction

It is relatively difficult to construct continuously-differentiable (C_1) finite elements. Most C_1-elements were constructed in the early 1970s, cf. [5]. The most famous C_1 element is the Argyris P_5-triangle [2]. The element was extended to the full C_1-P_5 space, known as the Morgan-Scott P_k-triangles, for all $k \geq 5$ [10]. In the other direction, we have the Bell reduced P_5-triangle [4].

There are two types of composite-triangular C_1 elements, the Hsieh-Clough-Tocher P_3-triangle [5] (subdividing each macro-triangle into 3 triangles), and Powell-Sabin P_2-triangle [13] (subdividing each macro-triangle into 6 triangles). There are many variances and extensions, such as, the reduced Hsieh-Clough-Tocher P_3-triangle [12], the Douglas-Dupont-Percell-Scott P_k-triangle [6], and the Powell-Sabin-Heindl P_2-triangle [8]. There is also another type of quadrilateral-based triangle elements (subdividing each quadrilateral into 4 triangles), the Fraeijs de Veubeke-Sander P_3-triangle and its reduced version [16, 7].

The most complicated construction would be the C_1-P_5 element on tetrahedral grids by Ženišek [24]. All above elements were created in early 1970s. But many extensions or expansions on the above C_1 finite elements were done in last few years. The Hsieh-Clough-Tocher P_3-triangle is extended to d space-dimensional C_m-P_k elements, cf. a recent work [20] and references therein. Similar extensions are made to the Powell-Sabin element too, cf. [19]. In particular, a 4D Powell-Sabin element is constructed explicitly in [21]. Also, the Fraeijs de Veubeke-Sander element is extend on triangulations based on rectangles or cubes, cf. [18]. On the general triangulation, the Ženišek 3D C_1-P_9 element is modified and extended to all polynomial degree $k \geq 9$ in [26].

In this work, we extend the 2D Argyris element and the 3D Ženišek element to a family of 4D C_1-P_k element, for all $k \geq 17$. By studying the constraints on the coefficients of polynomials

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in the Bézier-Bernstein form, Alfeld and Sirvent showed the existence of C_m-P_k superspline spaces in d space dimension when $k > m2^d$ in [1]. By the theory, 17 is the lowest polynomial degree in 4D. There is a small difference between constructing splines and finite elements, cf. for example, [17, 25]. In finite elements, explicit dual functionals and local basis functions are required. But a spline construction may or may not produce a local basis as it can define splines implicitly. In our construction, we could not use the full set of Lagrange nodes inside triangles, tetrahedra and 4D simplices, due to geometric constraints, cf Definition 2.1.3(c), 2.1.4(b) and 2.1.5. This feature does not appear in 2D and 3D C_1 elements. In particular, a family of special 3D C_2-P_k elements is obtained as a byproduct. Many families of 2D and 3D elements, of less interest, are generated too as restrictions of the 4D C_1 element in lower space dimensions.

There are some 4D partial differential equations of interest, [3, 14]. The C_1 finite elements are constructed for solving 4-th order elliptic equations, such as biharmonic equations, known as H_2 conforming elements. There is also a type of nonconforming element, the Morley element [11], for 2D biharmonic equation. This element is also extended to 3D, 4D and higher space dimensions [15, 22].

The rest of this manuscript is organized as follows. In Section 2, the 4D C_1-P_k finite element family is defined. In Section 3, a family of 2D C_4-P_k element is introduced for the analysis of the 4D element. In Section 4, a family of special 3D C_2-P_k element is analyzed. Finally in Section 5, the unisolvence of the 4D C_1 element is proven.

2 A family of 4D C_1-P_k finite elements

In this section, we define a family of 4D C_1 finite elements. A finite element is defined by a triple (P_K, K, Σ_K), where K is an element, P_K is a finite dimensional space of functions defined on K, and Σ is a basis for the linear functionals on P_K, cf. [5]. For the 4D C_1 finite element, K is a 4D simplex, P_K is the space of polynomials P_k, $k \geq 17$. We need to define Σ_K. This is done in Definition 2.1, where $\Sigma_{\hat{K}}$ is defined for the reference element $(P_{\hat{K}}, \hat{K}, \Sigma_{\hat{K}})$. The reference simplex is the unit right simplex at the origin:

$$\hat{K} = \left\{ (x_1, x_2, x_3, x_4) \mid 0 \leq x_i, \ \sum_i x_i \leq 1 \right\}. \tag{2.1}$$

For convenience, we denote the 5 vertices of $\hat{K}$ by

$$p_0 = \begin{pmatrix} 0\\0\\0\\0 \end{pmatrix}, \ p_1 = \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \ p_2 = \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \ p_3 = \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \ p_4 = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}.$$

Definition 2.1 Let the reference simplex $\hat{K}$ be defined by (2.1). For each edge on $\hat{K}$, denoted by $\mathbf{e}_i$, we choose arbitrarily three unit vectors orthogonal to the edge. Let $\mathbf{t}_j$ denote two unit normal vectors for each triangle of $\hat{K}$. $\mathbf{f}_l$ denote one normal vector on each tetrahedron of $\hat{K}$. The set of linear functionals, $\Sigma_{\hat{K}} = \{f\}$, is defined by the following nodal values and derivatives, $f(u)$, for any $u \in P_k(\hat{K}) = P_{\hat{K}}$.

1. Vertex: (Figure 1) The nodal values and derivatives up to order 8 at each of 5 vertices p_i: $\{D_j u(p_i), \quad j = 0, 1, ..., 8\}$.

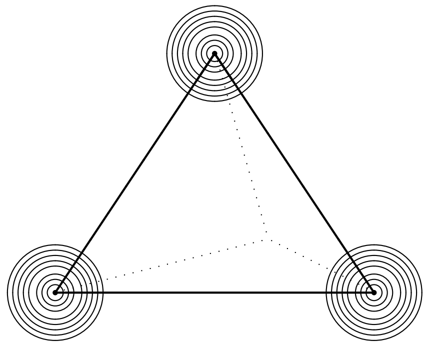

Figure 1: $D_i u(p_j),\ i \le 8$ at 5 vertices of $\hat{K}$ (Definition 2.1.1.)

2. Edge: (Figures 3 and 4)

   (a) The nodal values $\{u(q_i)\}$ at $(k-17)$ equally-distributed internal points, on each of 10 edges of $\hat{K}$: on edge $p_{i_1}p_{i_2}$,

$$q_i = \frac{i}{k-16}p_{i_1} + \frac{k-16-i}{k-16}p_{i_2}, \quad 1 \le i \le (k-17). \tag{2.2}$$

   (b) 3 first order normal derivatives $\{u_{\mathbf{e}_j}(q_i)\}$ at $(k-16)$ equally-distributed internal points, cf.(2.2), on each edge.

   (c) 6 second order normal derivatives $\{u_{\mathbf{e}_j\mathbf{e}_l}(q_i)\}$ at $(k-15)$ equally-distributed internal points, cf.(2.2), on each edge.

   (d) 10 third order normal derivatives $\{u_{\mathbf{e}_j\mathbf{e}_l\mathbf{e}_m}(q_i)\}$ at $(k-14)$ equally-distributed internal points, cf.(2.2), on each edge.

   (e) 15 fourth order normal derivatives $\{u_{\mathbf{e}_j\mathbf{e}_l\mathbf{e}_m\mathbf{e}_n}(q_i)\}$ at $(k-13)$ equally-distributed internal points, cf.(2.2), on each edge.

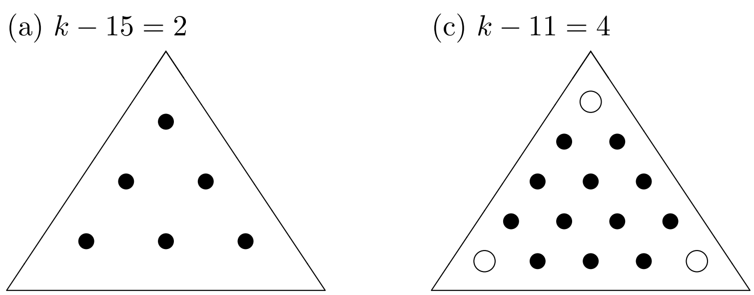

Figure 2: P_{k-15} nodes and P_{k-11} 2nd derivative nodes in Definition 2.1.2.(a) and (c).

3. Triangle: (Figure 5)

   (a) The nodal values $\{u(r_{ij})\}$ at the internal 2D Lagrange nodes for P_{k-15} polynomials (cf. [5]), on each of 10 triangles of $\hat{K}$: on triangle $p_{i_1}p_{i_2}p_{i_3}$, cf. Figure 2,

$$r_{ij} = \frac{i}{k-12}p_{i_1} + \frac{j}{k-12}p_{i_2} + \frac{k-12-i-j}{k-12}p_{i_3},$$
$$1 \le i \le (k-14), \quad 1 \le j < (k-13-i). \tag{2.3}$$

(b) 2 first order normal derivatives $\{u_{\mathbf{t}_j}(r_{ij})\}$ at the internal 2D Lagrange nodes for P_{k-13} polynomials on each triangle, cf. (2.3).

(c) 3 2nd order normal derivatives $\{u_{\mathbf{t}_j\mathbf{t}_l}(r_{ij})\}$ at the internal 2D Lagrange nodes for P_{k-11} polynomials, except the three corner nodes, on each triangle: on triangle $p_{i_1}p_{i_2}p_{i_3}$, cf. Figure 2,

$$r_{ij} = \frac{i}{k-8}p_{i_1} + \frac{j}{k-8}p_{i_2} + \frac{k-8-i-j}{k-8}p_{i_3}, \\ 1 \le i \le (k-10),\ 1 \le j < (k-9-i), \\ i+j \ne 2,\ i \ne (k-10),\ j \ne (k-10). \tag{2.4}$$

4. Tetrahedron: (Figure 6)

(a) The nodal values $\{u(s_{ijl})\}$ at the internal 3D Lagrange nodes for P_{k-12} polynomials, inside each of 5 tetrahedra of $\hat{K}$: on tetrahedron $p_{i_1}p_{i_2}p_{i_3}p_{i_4}$,

$$s_{ijl} = \frac{i}{k-8}p_{i_1} + \frac{j}{k-8}p_{i_2} + \frac{l}{k-8}p_{i_3} + \frac{k-8-i-j-l}{k-8}p_{i_4}, \\ 1 \le i \le (k-11),\ 1 \le j \le (k-10-i), \\ 1 \le l \le (k-9-i-j). \tag{2.5}$$

(b) One normal derivative $\{u_{\mathbf{f}_j}(s_{ijl})\}$ at the internal 3D Lagrange nodes for P_{k-9} polynomials, except the 4 corner nodes at each of the four vertices of the tetrahedron: on tetrahedron $p_{i_1}p_{i_2}p_{i_3}p_{i_4}$,

$$s_{ijl} = \frac{i}{k-5}p_{i_1} + \frac{j}{k-5}p_{i_2} + \frac{l}{k-5}p_{i_3} + \frac{k-5-i-j-l}{k-5}p_{i_4}, \\ 1 \le i \le (k-8),\ 1 \le j \le (k-7-i), \\ 1 \le l \le (k-6-i-j),\ i+j+k \ne 3, \\ i \ne (k-8),\ j \ne (k-8), l \ne (k-8). \tag{2.6}$$

5. 4D Simplex: (Figure 7) The nodal values $\{u(t_{ijlm})\}$ at the internal 4D Lagrange nodes for P_{k-10} polynomials, except 5 corner nodes at the 5 vertices. inside $\hat{K}$:

$$t_{ijlm} = \frac{i}{k-5}p_{i_1} + \frac{j}{k-5}p_{i_2} + \frac{l}{k-5}p_{i_3} + \frac{m}{k-5}p_{i_4} \\ + \frac{k-5-i-j-l-m}{k-5}p_{i_5}, \quad 1 \le i \le (k-9),\ 1 \le j \le (k-8-i), \\ 1 \le l \le (k-7-i-j),\ 1 \le m \le (k-6-i-j-l), \\ i+j+k+l \ne 4,\ \max\{i,j,l,m\} < (k-9). \tag{2.7}$$

∎

We show that the number of linear functionals in Σ is the same as the dimension of the space of 4D P_k polynomials.

Theorem 2.1 *Let $(P_{\hat{K}}, \hat{K}, \Sigma_{\hat{K}})$ be defined by Definition 2.1.*

$$\dim \Sigma_{\hat{K}} = \dim P_{\hat{K}}, \quad k \ge 17.$$

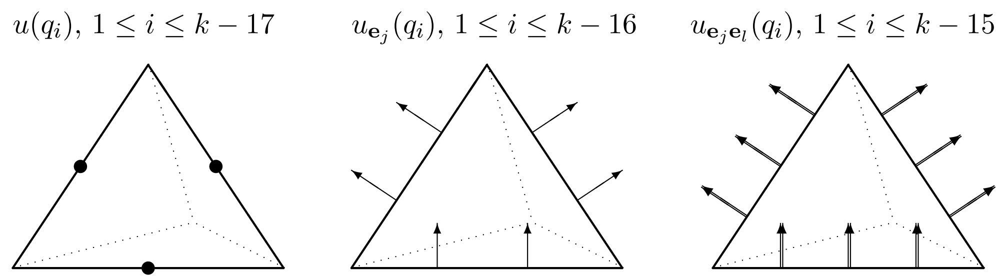

Figure 3: Degrees of freedom at 10 edges of $\hat{K}$ (Definition 2.1.2.(a)–(c)).

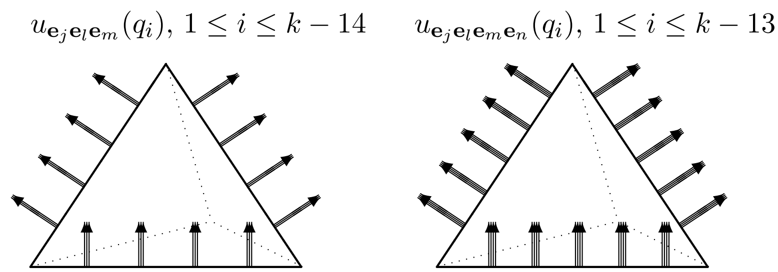

Figure 4: 3rd and 4th order normal derivatives at 10 edges of $\hat{K}$ (Definition 2.1.2.(d)–(e)).

Proof. For the space of 4D P_k polynomials, the dimension is

$$\begin{aligned}\dim P_k &= \frac{(k+1)(k+2)(k+3)(k+4)}{24} \\ &= \frac{1}{24}k^4 + \frac{5}{12}k^3 + \frac{35}{24}k^2 + \frac{25}{12}k + 1. \end{aligned} \tag{2.8}$$

Let us list the numbers of linear functionals in one item of Definition 2.1 by d_i. At 5 vertices of $\hat{K}$, the derivatives of orders from 0 to 8 are included in $\Sigma_{\hat{K}}$.

$$d_1 = 5 \dim P_8 = 5 \cdot \frac{(8+1)(8+2)(8+3)(8+4)}{24} = 2475.$$

At the 10 edges of $\hat{K}$, by Definition 2.1.2(a)–(e), we have

$$\begin{aligned} d_{2(a)} &= 10(k-17), \\ d_{2(b)} &= 10(3)(k-16), \\ d_{2(c)} &= 10(6)(k-15), \\ d_{2(d)} &= 10(10)(k-14), \\ d_{2(e)} &= 10(15)(k-13), \\ d_2 &= d_{2(a)} + d_{2(b)} + d_{2(c)} + d_{2(d)} + d_{2(e)} = 350k - 4900. \end{aligned}$$

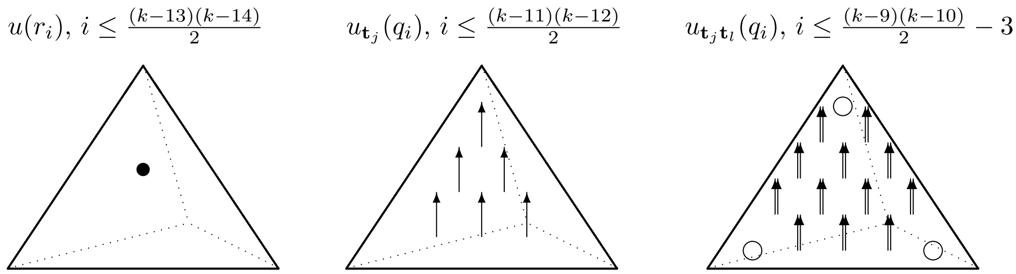

Figure 5: Degrees of freedom at 10 triangles of $\hat{K}$ (Definition 2.1.3.(a)–(c)).

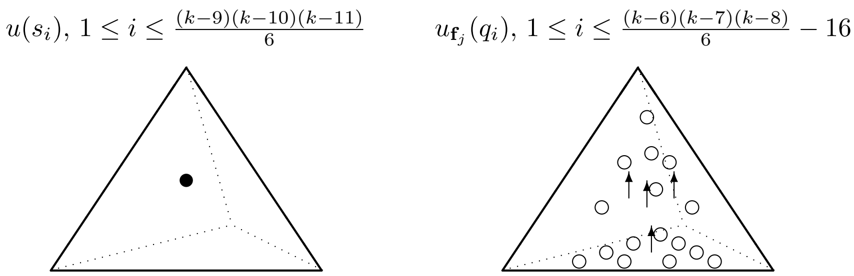

Figure 6: Degrees of freedom inside 5 tetrahedra of $\hat{K}$ (Definition 2.1.4.(a)–(b)).

On the 10 face triangles of $\hat{K}$, by Definition 2.1.3(a)–(c), we have

$$\begin{aligned}
d_{3(a)} =& 10 \dim P_{k-15,2} = (10)\frac{(k-13)(k-14)}{2}, \\
d_{3(b)} =& 10(2) \dim P_{k-13,2} = (20)\frac{(k-11)(k-12)}{2}, \\
d_{3(c)} =& 10(3)(\dim P_{k-11,2} - 3) = 30(\frac{(k-9)(k-10)}{2} - 3), \\
d_3 =& d_{3(a)} + d_{3(b)} + d_{3(c)} = 30k^2 - 650k + 3490.
\end{aligned}$$

Here we used notations $P_{k,2}$ for polynomials of degree k or less in 2 variables. On the 5 face tetrahedra of $\hat{K}$, by Definition 2.1.4(a)–(b), we have

$$\begin{aligned}
d_{4(a)} =& 5 \dim P_{k-12,3} = (5)\frac{(k-9)(k-10)(k-11)}{6}, \\
d_{4(b)} =& 5(\dim P_{k-9,3} - 16) = 5(\frac{(k-6)(k-7)(k-8)}{6} - 16), \\
d_4 =& d_{4(a)} + d_{4(b)} = \frac{5}{3}k^3 - \frac{85}{2}k^2 + \frac{2225}{6}k - 1185.
\end{aligned}$$

$$u(t_i), 1 \le i \le \frac{(k-6)(k-7)(k-8)(k-9)}{4!} - 5$$

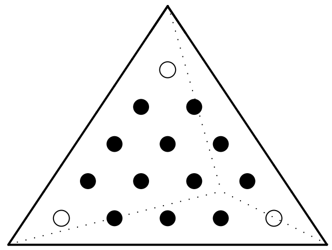

Figure 7: Degrees of freedom inside 4D simplex $\hat{K}$ (Definition 2.1.5.)

Finally, inside the 4D simplex $\hat{K}$, by Definition 2.1.5, we have

$$\begin{aligned} d_5 =& \dim P_{k-10} - 5 = \frac{(k-6)(k-7)(k-8)(k-9)}{24} - 5 \\ =& \frac{1}{24}k^4 - \frac{5}{4}k^3 + \frac{335}{24}k^2 - \frac{275}{4}k + 121. \end{aligned}$$

Summing up the numbers of linear functionals above, by (2.8), we proved the theorem as

$$\begin{aligned} & d_1 + d_2 + d_3 + d_4 + d_5 \\ =& 2475 + 350k - 4900 + 30k^2 - 650k + 3490 \\ & + \frac{5}{3}k^3 - \frac{85}{2}k^2 + \frac{2225}{6}k - 1185 + \frac{1}{24}k^4 - \frac{5}{4}k^3 + \frac{335}{24}k^2 - \frac{275}{4}k + 121 \\ =& \frac{1}{24}k^4 + \frac{5}{12}k^3 + \frac{35}{24}k^2 + \frac{25}{12}k + 1 = \dim P_k. \end{aligned}$$

∎

In order to show the uni-solvent and the continuous differentiability, we first study the spaces of the restriction of the 4D finite element functions on triangles and on tetrahedra. They form a 2D C_4-P_k space and a 3D C_2-P_k space.

3 A family of 2D C_4-P_k finite elements

We define a family of 2D C_4-P_k finite element spaces, $(P_{K_2}, K_2, \Sigma_{K_2})$, by restricting the finite element space (P_K, K, Σ_K) in Definition 2.1. We define and study this family of elements for the purpose of analyzing the 4D C_1-P_k element. These 2D C_4-P_k elements are not completely new. It is pointed in [23] that there exist 2D C_m spaces of degree $4m+1$ polynomials. When $m = 4$, we get 17, our lowest degree of polynomial in the family. In fact, 17 is not the optimal polynomial degree. By [9], $4m+1$ can be improved to $3m+2$, i.e, 2D C_4-P_{14} elements can be constructed.

Definition 3.1 (cf. Definition 2.1) Let the reference simplex $\hat{K}_2$ be the unit right triangle at the origin. Let $P_{\hat{K}_2}$ be the set of 2D polynomials of degree k, $k \ge 17$. Let the set of functionals $\Sigma_{\hat{K}_2} = \{f\}$ be defined by

1. Vertex: The nodal values and derivatives up to order 8 at each of 3 vertices $\hat{K}_2$.
2. Edge: (cf. Figures 3 and 4) $(k-17+i)$ normal derivative of order i, $0 \le i \le 4$, at internal points on the edge.
3. Triangle: (Figure 5) Nodal values at 2D P_{k-15} internal Lagrange nodes.

∎

Theorem 3.1 *The set of functional $\Sigma_{\hat{K}_2}$ in Definition 3.1 is a dual basis for P_k, i.e., for any $v \in P_k$ such that $f(v) = 0$ for all $f \in \Sigma_{\hat{K}_2}$, then $v = 0$.*

Proof. Let $v \in P_{\hat{K}_2} = P_k$, $k \ge 17$, where $\hat{K}_2 = \{(x_1, x_2) \mid 0 \le x_1, x_2, 1 - x_1 - x_2 \le 1\}$. We restrict $v(x_1, x_2)$ to one edge on $x_2 = 0$:

$$v_1(x_1) = v(x_1, 0).$$

1. Since $D_i(p_j) = 0$ at the two vertices of $x_2 = 0$ edge of $\hat{K}_2$, we get

   $$v_1^{(i)}(0) = v_1^{(i)}(1) = 0, \qquad 0 \le i \le 8,$$

   and $v_1(x_1) = x_1^9(1-x_1)^9 v_{1,k-18}(x_1)$ where $v_{1,k-18}(x_1)$ is a degree $(k-18)$ polynomial. Since $v(q_i) = 0$ on $(k-17)$ internal points q_i of each edge of $\hat{K}$ by Definition 3.1.2, we have

   $$v_1(x_1) = x_1^9(1-x_1)^9 \left(\prod_{i=1}^{k-17} (x_1 - \frac{i}{k-16}) \right) v_{1,-1}(x_1),$$

   where $v_{1,-1}(x_1)$ is a degree minus one polynomial, i.e., $p_{-1} \equiv 0$. Hence $v_1(x_1) = 0$ and

   $$v(x_1, x_2) = x_2 v_{2,1}(x_1, x_2),$$

   where $v_{2,1}(x_1, x_2)$ is a degree $(k-1)$ polynomial.

2. We next consider

   $$\frac{\partial}{\partial x_2} v(x_1, x_2) = v_{2,1}(x_1, x_2) + x_2 \frac{\partial}{\partial x_2} v_{2,1}(x_1, x_2).$$

   When it is restricted on the edge $x_2 = 0$, we let

   $$v_2(x_1) = \frac{\partial}{\partial x_2} v(x_1, 0).$$

   By Definition 3.1.1, we have all derivatives of v_2 zero up to order 7, and the following decomposition,

   $$v_2^{(i)}(0) = v_2^{(i)}(1) = 0, \ i \le 7, \qquad \text{and } v_2 = x_1^8(1-x_1)^8 v_{2,k-17}(x_1),$$

   for some degree $(k-17)$ degree polynomial $v_{2,k-17}(x_1)$. By Definition 3.1.2,

   $$v_2(x_1) = x_1^8(1-x_1)^8 \left(\prod_{i=1}^{k-16} (x_1 - \frac{i}{k-15}) \right) v_{2,-1}(x_1).$$

   Hence

   $$v_2(x_1) \equiv 0 \ \text{ and } \ v(x_1, x_2) = x_2^2 v_{2,2}(x_1, x_2).$$

3. Repeating the second step above, by Definition 3.1.2, we would obtain

$$v(x_1, x_2) = x_2^5 v_{2,5}(x_1, x_2),$$

where $v_{2,5}(x_1, x_2)$ is a degree $(k-5)$ polynomial. Symmetrically, we can restrict $v_2(x_1, x_2)$ on the other two edges, $x_2 = 0$ and $x_1 + x_2 = 1$ to get

$$v(x_1, x_2) = x_1^5 x_2^5 (1 - x_1 - x_2)^5 v_{2,15}(x_1, x_2),$$

where $v_{2,15}(x_1, x_2)$ is a degree $(k - 15)$ polynomial.

4. By Definition 3.1.3, $v_{2,15}(x_1, x_2)$ is zero at all internal Lagrange nodes for 2D P_{k-15} polynomials. Hence $v_{2,15}(x_1, x_2) = 0$ and

$$v(x_1, x_2) = 0.$$

The proof is completed. ∎

Via affine mappings, it is standard to show that Σ_{K_2} is uni-solvent on a general triangle K_2, cf. [5]. Now, on any 2D regular triangulation, the finite element space $(P_{K_2}, K_2, \Sigma_{K_2})$ is continuous differentiable up to order 4. It is easy to see this because such a finite element function has the same normal derivatives on two sides of each edge up to order 4. So the function has all order 4 derivatives continuous on the edge, and the function is in C_4.

4 A family of 3D C_2-P_k finite elements

In this section, we define a family of C_2-P_k polynomials on tetrahedral grids. On each tetrahedron such a function is a restriction of a 4D C_1-P_k function defined in Definition 2.1. However, we have "missing" values in the set of functionals in Definition 4.1.3, cf. Figure 2 and Definition 2.1.3.(c). Some nonstandard analysis is needed.

Definition 4.1 (cf. Definition 2.1) Let the reference simplex $\hat{K}_3$ be the unit right tetrahedron at the origin. Let $P_{\hat{K}_3}$ be the set of 3D polynomials of degree k, $k \geq 17$. Let the set of functionals $\Sigma_{\hat{K}_3} = \{f\}$ be defined by

1. Vertex: The nodal values and derivatives up to order 8 at each of 3 vertices of $\hat{K}_2$.

2. Edge: (cf. Figures 3 and 4) All $(k - 17 + i)$ normal derivatives of order $i = 0, 1, 2, 3, 4$, at internal points on the edge.

3. Triangle: (Figures 2 and 5) Order $(i = 0, 1, 2)$ normal derivative at 2D $P_{k-15+2i}$ internal Lagrange nodes on each face triangle, except 3 2nd order normal derivatives at the three corner nodes, cf. Figure 2.

4. Tetrahedron: (Figure 6) The nodal values at the internal 3D Lagrange nodes for P_{k-12} degree polynomials inside $\hat{K}_3$.

∎

Theorem 4.1 *The set of functional $\Sigma_{\hat{K}_3}$ in Definition 4.1 is a dual basis for P_k, i.e., for any $v \in P_k$ such that $f(v) = 0$ for all $f \in \Sigma_{\hat{K}_3}$, then $v = 0$.*

Proof. Let $v \in P_{\hat{K}_3} = P_k$, $k \ge 17$, where

$$\hat{K}_3 = \{(x_1, x_2, x_3) \mid 0 \le x_1, x_2, x_3, 1 - x_1 - x_2 - x_3 \le 1\}.$$

For the function $v(x_1, x_2, x_3)$ satisfying $f(v) = 0$ for all $f \in \Sigma_{\hat{K}_3}$, we restrict it to one triangle on $x_3 = 0$:

$$v_1(x_1, x_2) = v(x_1, x_2, 0).$$

1. $v_1(x_1, x_2)$ satisfies all conditions in Definition 3.1. By Theorem 3.1, $v_1(x_1, x_2) = 0$ and

$$v(x_1, x_2, x_3) = x_3 v_{2,1}(x_1, x_2, x_3),$$

   where $v_{2,1}(x_1, x_2, x_3)$ is a degree $(k-1)$ polynomial.

2. $v_{2,1}(x_1, x_2, x_3)$ is same as $\partial v / \partial x_3$ on the triangle $x_3 = 0$. Repeating the proof for Theorem 3.1 for function $\partial v / \partial x_3$ on $x_3 = 0$, it follows that

$$v_{2,1}(x_1, x_2, 0) = x_1^4 x_2^4 (1 - x_1 - x_2)^4 v_{2,13}(x_1, x_2),$$

   where $v_{2,13}(x_1, x_2)$ is a degree $(k-13)$ polynomial. By the second order normal derivatives in Definition 4.1.3,

$$v_{2,1}(x_1, x_2, 0) = 0, \quad \text{and} \quad v(x_1, x_2, x_3) = x_3^2 v_{2,2}(x_1, x_2, x_3), \tag{4.1}$$

   where $v_{2,2}(x_1, x_2, x_3)$ is a degree $(k-2)$ polynomial.

3. Repeating the proof for Theorem 3.1, or the Step 2 above, it follows that

$$v_{2,2}(x_1, x_2, 0) = x_1^3 x_2^3 (1 - x_1 - x_2)^3 v_{2,11}(x_1, x_2),$$

   where $v_{2,11}(x_1, x_2)$ is a degree $(k-11)$ polynomial. However, different from all previous cases, here we do not have all 2nd order normal values zero for $v(x_1, x_2, x_3)$ at the 2D P_{k-11} nodes. We have 3 corner nodes missing in Definition 4.1.3. What we have is, cf. (2.4),

$$v_{2,2}(x_1, x_2, 0) = x_1^3 x_2^3 (1 - x_1 - x_2)^3 \sum_{j=1}^{3} c_j \phi_j(x_1, x_2), \tag{4.2}$$

   where ϕ_i are three basis functions for P_{k-11}:

$$\begin{aligned}
\phi_1(x_1, x_2) &= \prod_{j=1}^{k-12} \left(x_1 - \frac{j}{k-8} \right), \\
\phi_2(x_1, x_2) &= \prod_{j=1}^{k-12} \left(x_2 - \frac{j}{k-8} \right), \\
\phi_3(x_1, x_2) &= \prod_{j=1}^{k-12} \left(1 - x_1 - x_2 - \frac{j}{k-8} \right).
\end{aligned}$$

We go back to Definition 4.1.1 to get that

$$\begin{aligned}
\frac{\partial^8}{\partial {x_1}^4 \partial {x_2}^4} v(0,0,0) &= 324 \frac{\partial^2}{\partial x_1 \partial x_2} \sum_{j=1}^{3} (1-x_1-x_2)^3 c_j \phi_j(x_1,x_2) \Big|_{x_1=0, x_2=0} \\
&= 324 \sum_{j=1}^{3} \Big(6 c_j \phi_j(0,0) - 9 c_j \frac{\partial}{\partial x_1} \phi_j(0,0) \\
&\qquad - 9 c_j \frac{\partial}{\partial x_2} \phi_j(0,0) + c_j \frac{\partial^2}{\partial x_1 \partial x_2} \phi_j(0,0) \Big) \\
&= 0.
\end{aligned}$$

This leads us to a linear equation:

$$c_1 + c_2 + c_3 - \frac{3}{2}\alpha(c_1 + c_2 - 2c_3) - c_3\beta = 0, \tag{4.3}$$

where the constants are

$$\begin{aligned}
\alpha &= (k-8) \sum_{i=1}^{k-12} \frac{1}{i} = (k-8)\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k-12}\right), \\
\beta &= \frac{k-8}{6} \sum_{i=1}^{k-12} \sum_{j\neq i,\ j=1}^{k-12} \frac{1}{i \cdot j} = \frac{k-8}{6}\left(\frac{1}{1\cdot 2} + \frac{1}{1\cdot 3} + \cdots + \frac{1}{2\cdot 3} + \cdots\right).
\end{aligned}$$

By the other two conditions in Definition 4.1.1,

$$\frac{\partial^8}{\partial {x_1}^4 \partial {x_2}^4} v(1,0,0) = 0, \quad \frac{\partial^8}{\partial {x_1}^4 \partial {x_2}^4} v(0,1,0) = 0,$$

or, to be precise, by these two conditions,

$$\frac{\partial^8}{\partial {x_1}^4 \partial y^4} v(1,0,0) = 0, \quad \frac{\partial^8}{\partial y^4 \partial {x_2}^4} v(0,1,0) = 0,$$

where

$$y = 1 - x_1 - x_2, \tag{4.4}$$

we get the other two versions of equation (4.3):

$$c_1 + c_2 + c_3 - \frac{3}{2}\alpha(c_3 + c_2 - 2c_1) - c_1\beta = 0, \tag{4.5}$$

$$c_1 + c_2 + c_3 - \frac{3}{2}\alpha(c_3 + c_1 - 2c_2) - c_2\beta = 0. \tag{4.6}$$

Subtracting (4.6) from (4.5), it follows $c_1 = c_2$. Then, by (4.3) and (4.5), we get $c_1 = c_2 = c_3$. Replacing c_2 and c_1 by c_3 in (4.6), we get $c_3 = 0$. Thus $c_1 = c_2 = c_3 = 0$. By (4.2), $v_{2,2}(x_1, x_2, 0) = 0$. Therefore, by (4.1),

$$v(x_1, x_2, x_3) = x_3^3 v_{2,3}(x_1, x_2, x_3),$$

where $v_{2,3}(x_1, x_2, x_3)$ is some degree $(k-3)$ polynomial.

4. By symmetry, it follows that

$$v(x_1,x_2,x_3) = x_1^3x_2^3x_3^3(1-x_1-x_2-x_3)^3 v_{2,12}(x_1,x_2,x_3),$$

where $v_{2,12}(x_1,x_2,x_3)$ is some degree $(k-12)$ polynomial. Applying Definition 4.1.4, we get

$$v_{2,12}(x_1,x_2,x_3) = 0, \quad \text{and} \quad v(x_1,x_2,x_3) = 0.$$

The proof is completed. ∎

The proof above carries over to the general tetrahedron K. We only need to change partial derivatives, such as changing $\partial_{x_1}v$ to $\partial_{\mathbf{e}_1}v$ where $\mathbf{e}_1$ is a normal vector on an edge or on a triangle, similar to (4.4). Therefore $(P_{K_3}, P_k, \Sigma_{K_3})$ is well defined. It is then standard to verify that $(P_{K_3}, P_k, \Sigma_{K_3})$ defines a C_1 space on a regular tetrahedral grid.

5 Uni-solvent and differentiability

We will show the uni-solvent of the set of linear functionals $\Sigma_{\hat{K}}$ on $P_{\hat{K}}$, defined in Definition 2.1. We further show that the finite element functions of piecewise polynomials on simplicial grids are continuously differentiable, when defined by Definition 2.1.

Theorem 5.1 *The set of functional $\Sigma_{\hat{K}}$ in Definition 2.1 is a dual basis for P_k, i.e., for any $v \in P_k$ such that $f(v) = 0$ for all $f \in \Sigma_{\hat{K}}$, then $v = 0$.*

Proof. Let $v \in P_{\hat{K}} = P_k$ and $f(v) = 0$ for all $f \in \Sigma_{\hat{K}}$.

1. Restricting v on a face tetrahedron $x_4 = 0$ of the reference simplex $\hat{K}$, the restriction satisfies $f(v) = 0$ for all $f \in \Sigma_{K_3}$, defined in Definition 4.1. By Theorem 4.1, the restriction is 0. So
$$v(x_1,x_2,x_3,x_4) = x_4 v_{0,1}(x_1,x_2,x_3,x_4), \tag{5.1}$$
where $v_{0,1}$ is a degree $(k-1)$ polynomial.

2. Next, we study $\partial v/\partial x_4$:
$$v_{1,1}(x_1,x_2,x_3) = \left.\frac{\partial v}{\partial x_4}\right|_{x_4=0} = v_{0,1}(x_1,x_2,x_3,0).$$
   (a) By Definition 2.1.1, 2(b)-(e), and 3(b)-(c), we obtain
   $$v_{1,1}(x_1,x_2,x_3) = x_3 v_{1,2}(x_1,x_2,x_3) \tag{5.2}$$
   where $v_{1,2}$ is a degree $(k-2)$ polynomial.

   (b) To derive another factor of x_3, we let
   $$v_{2,2}(x_1,x_2) = \left.\frac{\partial^2 v}{\partial x_3 \partial x_4}\right|_{x_3=0,x_4=0} = v_{1,2}(x_1,x_2,0).$$

But slightly different from Step 2(b) above, we have three corner values missing, cf Definition 2.1.3(c). We need to adopt the technique in (4.2)–(4.6). This time, we use the conditions

$$\frac{\partial^8}{\partial {x_1}^3 \partial {x_2}^3 \partial x_3 \partial x_4} v(0,0,0,0) = 0, \qquad \frac{\partial^8}{\partial {x_1}^3 \partial {x_2}^3 \partial x_3 \partial x_4} v(1,0,0,0) = 0,$$
$$\frac{\partial^8}{\partial {x_1}^3 \partial {x_2}^3 \partial x_3 \partial x_4} v(0,1,0,0) = 0.$$

Similarly, we get

$$v_{1,2}(x_1,x_2,x_3) = x_3^2 v_{1,3}(x_1,x_2,x_3)$$

where $v_{1,3}$ is a degree $(k-3)$ polynomial.

(c) By symmetry, we get

$$v_{1,2}(x_1,x_2,x_3) = x_1^2 x_2^2 x_3^2 (1-x_1-x_2-x_3)^2 v_{1,9}(x_1,x_2,x_3)$$

where $v_{1,9}$ is a degree $(k-9)$ polynomial.

(d) By Definition 2.1.4(b), we could not derive immediately $v_{1,9} = 0$, as we have 4 corner values missing. Similar to (4.2)–(4.6), we have 4 dimensional degrees of freedom to be determined,

$$v_{1,2}(x_1,x_2,x_3) = x_1^2 x_2^2 x_3^2 (1-x_1-x_2-x_3)^2 \sum_{i=1}^{4} c_{3,i} \phi_{3,i}(x_1,x_2,x_3),$$

where

$$\phi_{3,i}(x_1,x_2,x_3) = \prod_{j=1}^{k-10} \left(x_i - \frac{j}{k-5} \right), \qquad i = 1,2,3,$$
$$\phi_{4,i}(x_1,x_2,x_3) = \prod_{j=1}^{k-10} \left(1 - x_1 - x_2 - x_3 - \frac{j}{k-5} \right).$$

By the conditions in Definition 2.1.1, for example, by

$$\frac{\partial^8}{\partial {x_1}^3 \partial {x_2}^2 \partial {x_3}^2 \partial x_4} v(0,0,0,0) = 0,$$

we get

$$8(-c_{3,1} - c_{3,2} - c_{3,3} - c_{3,4} + (c_{3,1} - c_{3,4})\gamma) = 0, \tag{5.3}$$

where

$$\gamma = \prod_{j=1}^{k-10} \left(-\frac{k-5}{j} \right).$$

By symmetry, we can get the other three equations like (5.3). By the difference of such two equations, we see that the four constants are the same. By substituting $c_{3,i} = c_{3,1}$ in (5.3), it follows that all $c_{3,1} = 0$. Thus,

$$v_{1,9}(x_1,x_2,x_3) = 0, \quad \text{and} \quad v(x_1,x_2,x_3,x_4) = x_4^2 v_{0,2}(x_1,x_2,x_3,x_4), \tag{5.4}$$

where $v_{0,2}$ is a degree $(k-2)$ polynomial.

3. Symmetrically, we would extend (5.4) to

$$v(x_1,x_2,x_3,x_4) = x_1^2x_2^2x_3^2x_4^2(1-x_1-x_2-x_3-x_4)^2 v_{0,10}(x_1,x_2,x_3,x_4),$$

where $v_{0,10}$ is a degree $(k-10)$ polynomial. However, Definition 2.1.5 specifies 5 corner points less for the P_{k-10} internal Lagrange nodes. So we have to write

$$v(x_1,x_2,x_3,x_4) = x_1^2x_2^2x_3^2x_4^2(1-x_1-x_2-x_3-x_4)^2 \sum_{i=1}^{5} c_{4,i}\phi_{4,i}(x_1,x_2,x_3,x_4),$$

where

$$\phi_{4,i}(x_1,x_2,x_3,x_4) = \prod_{j=1}^{k-15}\left(x_i - \frac{j}{k-5}\right), \qquad i=1,2,3,4,$$

$$\phi_{5,i}(x_1,x_2,x_3,x_4) = \prod_{j=1}^{k-15}\left(1-x_1-x_2-x_3-x_4 - \frac{j}{k-5}\right).$$

By condition

$$\frac{\partial^8}{\partial {x_1}^2 \partial {x_2}^2 \partial {x_3}^2 \partial {x_4}^2} v(0,0,0,0) = 0,$$

we get

$$c_{4,5} = 0.$$

Symmetrically, we have all $c_{4,i} = 0$ and $v = 0$.

The proof is completed. ∎

The proof for Theorem 5.1 remains the same for a general 4D simplex K, as we did not use any special property of the reference $\hat{K}$ except the simpler notations. It is straightforward to show that (K, P_K, Σ_K) defines a C_1 space on a regular 4D simplicial grid. Since the interpolation operator specified by Σ_K preserves P_k polynomials, the finite element space has the optimal order of approximation, cf. [5].

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