de Rham complexes for weak Galerkin finite element spaces

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\begin{abstract}
Two de Rham complex sequences of the finite element spaces are introduced for weak finite element functions and weak derivatives developed in the weak Galerkin (WG) finite element methods on general polyhedral elements. One of the sequences uses polynomials of equal order for all the finite element spaces involved in the sequence and the other one uses polynomials of naturally descending orders. It is shown that the diagrams in both de Rham complexes commute for general polyhedral elements. The exactness of one of the complexes is established for the lowest order element.
\end{abstract}

1. Introduction

Finite element exterior calculus is a framework that explores the structural properties of finite element approximating spaces and their connection with basic differential operators such as gradient, curl, and divergence operators in differential calculus. Such differential operators usually form the basis of mathematical models for physical, engineering, and biological problems. The classical conforming finite element exterior calculus has provided good guidance in the design and analysis of various finite element methods for solving partial differential equations (PDE). In particular, the framework has proved to be powerful in analyzing well-posedness of finite element discretizations and revealing new finite elements for solving various PDE modeling problems arising from science and engineering. Finite element exterior calculus has been well developed for conforming finite element methods on simplicial elements, and they often appear in the form of de Rham complexes in the application of numerical solutions for PDEs [1–14].

In the last two decades, an extensive research effort has been made by the computational mathematics community in the development of finite element methods with discontinuous approximating functions. The weak Galerkin (WG) finite element method, first introduced in [15], is one of the recently developed finite element techniques based on discontinuous finite element functions. The essence of weak Galerkin finite element methods is the use of weak finite

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\textsuperscript{1} The research of Chunmei Wang was partially supported by National Science Foundation, United States Award DMS-1849483.

\textsuperscript{2} The research of Junping Wang was supported by the NSF IR/D program, United States, while working at National Science Foundation. However, any opinion, finding, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

\textsuperscript{3} This research was supported in part by National Science Foundation, United States Grant DMS-1620016.

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element functions and their weak derivatives defined as discrete distributions in polynomial subspaces. In general, weak Galerkin finite element formulations for partial differential equations can be derived naturally by replacing usual derivatives by discrete weak derivatives in the corresponding variational forms, with the option of adding stabilization term(s) to enforce a weak continuity of the approximating functions. Like most discontinuous finite element methods, WG method is applicable for finite element partitions with arbitrary shape of polygons/polyhedra.

The purpose of this paper is to present two de Rham diagrams for weak Galerkin finite element spaces with weakly defined differential operators. In particular, two finite element differential complexes are developed in the WG context on polyhedral elements. It is proved that the diagrams in the two de Rham complexes commute with simple, yet powerful $L^2$ projection operators from the continuous functionals space to the corresponding WG finite element “subspaces”.

**WG de Rham Complex 1:** Polynomials of Equal Order

For a given non-negative integer $k \geq 0$, we introduce four vector spaces $V^{(i)}_k(T)$, $i = 1, \ldots, 4$, that allow the operator operations shown in Fig. 1.1.

**WG de Rham Complex 2:** Polynomials of Descending Order $k \geq 3$

The above two WG de Rham complexes have some unique features; (1) they both work for discontinuous approximation on general polyhedral elements; (2) all the cochain projections $Q_h^{(i)}$ in Fig. 1.1 and $R_h^{(i)}$ in Fig. 1.2 are the standard $L^2$ projections; and (3) weakly defined differential operators are employed first time in finite element differential complexes. It should be noted that the weak gradient $V_{w}$, weak curl $V_{w} \times$ and weak divergence $V_{w}$ in the WG de Rham complexes 1 and 2 have been employed for solving various partial differential equations in many existing literatures, including [16–18]. The WG de Rham complexes 1 and 2 additionally provide discrete weak differential operators defined on surfaces, which is of great interest to the development of numerical methods for PDEs on surfaces or manifolds in general.

### 2. WG de Rham complexes

Let $T$ be a shape-regular polyhedron in the sense of [18]. Denote by $F(T)$, $E(T)$ and $V(T)$ the set of faces, edges, and vertices of $T$, respectively. For any face $f \in F(T)$, let $n$ be a unit normal vector to $f$ and $t$ be a unit tangential vector on $e \subset \partial f$ which obey the right-hand rule. Throughout this paper, we adopt the notation of $(\cdot, \cdot)_D$ for the $L^2$ inner product in $L^2(D)$, where $D$ could be a volume, a surface, or a curve.

#### 2.1. WG de Rham complex for polynomials of equal order

For a given non-negative integer $k \geq 0$, we introduce four vector spaces $V^{(1)}_k(T)$, $V^{(2)}_k(T)$, $V^{(3)}_k(T)$ and $V^{(4)}_k(T)$ that allow the operator operations shown as in Fig. 1.1. The first space $V^{(1)}_k(T)$ is defined by

$$V^{(1)}_k(T) = \{ v = \{ v_0, v_f, v_e, v_n \} : v_0 \in P_k(T), \ v_f \in P_k(f), \ v_e \in P_k(e), \ v_n \in \mathbb{R}, \ f \in F(T), \ e \in E(T), \ n \in V(T) \}.$$  \hspace{2cm} (2.1)
The second space $V^{(2)}_k(T)$ is given by
\[ V^{(2)}_k(T) = \{ u = \{ u_0, u_f, u_e \}, u_0 \in V^{(2)}_{k,0}(T), u_f \in V^{(2)}_{k,f}(f), \]
\[ u_e \in V^{(2)}_{k,e}(e), f \in F(T), e \in E(T) \}, \]
where
\[ V^{(2)}_{k,0}(T) = [P_k(T)]^3, \]
\[ V^{(2)}_{k,f}(f) = \{ u_f = u_1 t_{f,1} + u_2 t_{f,2} : u_1, u_2 \in P_k(f) \}, \]
\[ V^{(2)}_{k,e}(e) = \{ u_e = u t_e : u \in P_k(e) \}, \]
\[ t_{f,1} \text{ and } t_{f,2} \text{ in (2.4) are two orthogonal unit vectors tangential to } f, t_e \text{ in (2.5) is a tangent vector on } e. \] 

The third space $V^{(3)}_k(T)$ is defined as
\[ V^{(3)}_k(T) = \{ w = \{ w_0, w_f \}, w_0 \in V^{(3)}_{k,0}(T), w_f \in V^{(3)}_{k,f}(f), f \in F(T) \}, \]
where
\[ V^{(3)}_{k,0}(T) = [P_k(T)]^3, \]
\[ V^{(3)}_{k,f}(f) = \{ w_f = w_f n_f : w_f \in P_k(f) \}, \]
\[ w \text{ being a unit normal vector to the surface } f. \]

Finally, for any \( w = \{ w_0, w_f \} \in V^{(3)}_k(T) \), we define its weak divergence \( \nabla \cdot w \in V^{(4)}_k(T) \) by the following equation:
\[ (\nabla \cdot w, \tau)_T = -\langle w_0, \nabla \tau \rangle_T + \langle w_f, \tau n_f \rangle_T \quad \forall \tau \in V^{(4)}_k(T). \]
2.2. WG de Rham complex for polynomials of descending order

For a given integer $k \geq 3$, we introduce four polynomial spaces with descending orders for the diagram in Fig. 1.2: $W_k^{(1)}(T), W_k^{(2)}(T), W_k^{(3)}(T)$ and $W_k^{(4)}(T)$. The first polynomial space $W_k^{(1)}(T)$ is given by

$$W_k^{(1)}(T) = \{ v = \{ v_0, v_T, v_e, v_n \} : v_0 \in P_k(T), \quad v_T \in P_{k-1}(f), \quad v_e \in P_{k-2}(e), \quad v_n \in \mathbb{R}, \quad f \in F(T), \quad e \in E(T), \quad n \in V(T) \}.$$  \hfill (2.18)

The second space $W_k^{(2)}(T)$ is given by

$$W_k^{(2)}(T) = \{ u = \{ u_0, u_T, u_e \} : u_0 \in W_{k-1,0}(T), \quad u_T \in W_{k-2}(f), \quad u_e \in W_{k-3,e}(e), \quad f \in F(T), \quad e \in E(T) \},$$  \hfill (2.19)

where

$$W_{k-1,0}(T) = \{ P_{k-1}(T) \}^3,$$

$$W_{k-2}(f) = \{ u_T = u_1 t_{f,1} + u_2 t_{f,2} : u_1, u_2 \in P_{k-2}(f) \},$$

$$W_{k-3,e}(e) = \{ u_e = u_T n_e : u_T \in P_{k-3}(e) \}.$$  \hfill (2.20, 2.21, 2.22)

The third space $W_k^{(3)}(T)$ is defined as

$$W_k^{(3)}(T) = \{ w = \{ w_0, w_T \} : w_0 \in W_{k-2,0}(T), \quad w_T \in W_{k-3,f}(f) \},$$  \hfill (2.23)

where

$$W_{k-2,0}(T) = \{ P_{k-2}(T) \}^3,$$

$$W_{k-3,f}(f) = \{ w_T = w_T n_f : w_T \in P_{k-3}(f) \}.$$  \hfill (2.24, 2.25)

with $n_f$ a unit normal vector to the face $f$. The fourth polynomial space $W_k^{(4)}(T)$ is defined as follows:

$$W_k^{(4)}(T) = P_{k-3}(T).$$  \hfill (2.26)

Analogously, we define three weak gradient operators $\nabla_{w,0}$, $\nabla_{w,f}$ and $\nabla_{w,e}$ on $T$, $f$ and $e$ accordingly. More precisely, for $v = \{ v_0, v_T, v_e, v_n \} \in W_k^{(1)}(T)$, the weak gradient operator on volume $T$, denoted as $\nabla_{w,0} v \in W_{k-1,0}(T)$, is defined by

$$\langle \nabla_{w,0} v, \varphi \rangle_T = -(v_0, \nabla \cdot \varphi)_T + (v_T, \varphi \cdot n_{0T}) \quad \forall \varphi \in W_k^{(2)}(T).$$  \hfill (2.27)

The surface weak gradient operator on face $f \in F(T)$, denoted as $\nabla_{w,f} v \in W_{k-2,0}(f)$, is defined on $f$ satisfying

$$\langle \nabla_{w,f} v, \theta \times n_f \rangle_f = -(v_e, \nabla \cdot (\theta \cdot n_f))_f + (v_e, \theta \cdot t_{fj})_f \quad \forall \theta \in W_{k-2,0}(f).$$  \hfill (2.28)

The edge weak gradient or directional derivative on edge $e \in E(T)$, denoted as $\nabla_{w,e} v \in W_{k-3,0}(e)$, is defined on $e$ such that

$$\langle \nabla_{w,e} v, \varphi e \rangle_e = -(v_e, \nabla \cdot (\varphi e))_e + (v_n, \varphi e \cdot n_{ne})_e \quad \forall \varphi e \in W_{k-3,0}(e).$$  \hfill (2.29)

Collectively, we define a weak gradient mapping $\nabla_w : W_k^{(1)}(T) \mapsto W_k^{(2)}(T)$ as follows:

$$\nabla_w v := \{ \nabla_{w,0} v, \nabla_{w,f} v, \nabla_{w,e} v \}.$$  \hfill (2.30)

Next, we define two weak curl operators $\nabla_{w,0} \times$ and $\nabla_{w,f} \times$ on $T$ and $f$ respectively. For any $u = \{ u_0, u_f \} \in W_k^{(2)}(T)$, the weak curl on volume $T$, denoted as $\nabla_{w,0} \times u \in W_{k-2,0}(T)$, is defined on $T$ by

$$\langle \nabla_{w,0} \times u, \varphi \rangle_T = -(u_0, \nabla \times \varphi)_T + (u_f, \varphi \cdot n_{0T})_T \quad \forall \varphi \in W_{k-2,0}(T).$$  \hfill (2.31)

The surface weak curl on face $f \in F(T)$, denoted as $\nabla_{w,f} \times u \in W_{k-3,0}(f)$, is defined on $f$ by

$$\langle \nabla_{w,f} \times u, \tau n_f \rangle_f = -(u_f, \nabla \times \tau n_f)_f + (u_f, \tau f)_{fj} \quad \forall \tau n \in W_{k-3,0}(f).$$  \hfill (2.32)

The composite weak curl mapping $\nabla_w \times : W_k^{(2)}(T) \mapsto W_k^{(3)}(T)$ is then defined by

$$\nabla_w \times u := \{ \nabla_{w,0} \times u, \nabla_{w,f} \times u \}.$$  \hfill (2.33)

Finally, the weak divergence $\nabla_w \cdot w \in W_k^{(4)}(T)$ for any $w = \{ w_0, w_f \} \in W_{k-2}^{(3)}$ is defined on $T$ by the following equation

$$\langle \nabla_w \cdot w, \varphi \rangle_T = -(w_0, \nabla \varphi)_T + (w_f, \varphi n)_{fT} \quad \forall \varphi \in W_k^{(4)}(T).$$  \hfill (2.34)
3. Complex sequences

The goal of this section is to show that the WG sequences in Figs. 1.1 and 1.2 are complexes in the sense that the composition of any two consecutive operations is zero.

**Lemma 3.1.** For the weak gradient operator $\nabla_w$ and the weak curl operator $\nabla_w \times$ defined in (2.13) and (2.16) or in (2.30) and (2.33), the following identity holds true

$$\nabla_w \times (\nabla_w v) = 0$$

for all $v$ in $V^{(1)}_k(T)$ or $W^{(1)}_k(T)$.

**Proof.** By the definition of $\nabla_w \times$ in (2.16) or (2.33), we need to show $\nabla_w,0 \times (\nabla_w v) = 0$ and $\nabla_w,f \times (\nabla_w v) = 0$.

For any $v$ in $V^{(1)}_k(T)$ or $W^{(1)}_k(T)$, we have $\nabla_w v = (\nabla_w,0 v, \nabla_w,f v, \nabla_w,e v)$. By letting $u = \nabla_w v$ in (2.14) or (2.31) and using (2.10), (2.11), (2.27) and (2.28), we have for any $\theta$ in $V^{(1)}_{k,0}(T)$ or $W^{(1)}_{k,0}(T)$ that

$$[\nabla_w,0, \nabla w, \theta]_T = - (v_0, \nabla \cdot (\nabla w \cdot \theta))_T + (v_\gamma, \nabla \cdot \theta)_T + (\nabla w,0 v, \nabla w,e v)\theta_T$$

where we have used the fact that $\sum_{f \in F(T)} (v_\gamma, \theta)_f = 0$, as each edge is shared by two adjacent faces with tangential vector $t$ of opposite directions.

Next we show that $\nabla w,f \times (\nabla w v) = 0$ on each face $f \in F(T)$. To this end, on any given face $f \in F(T)$, by letting $u = \nabla_w v$ in (2.15) or (2.32) and using (2.11), (2.12), (2.28) and (2.29), we have for any $\tau$ in $V^{(2)}_{k,f}(T)$ or $W^{(2)}_{k,f}(T)$,

$$[\nabla w,f \times (\nabla w v), \tau n_f]_T = - (v_\gamma, \nabla \cdot \tau \cdot n_f)_T + (v_\gamma, \nabla w,e v, \tau n_f)_T$$

which implies (3.2) and thus completes the proof of the lemma.  

**Lemma 3.2.** For the weak curl and the weak divergence operator $\nabla w \times$ and $\nabla w \cdot$ defined in (2.16) and (2.17) or (2.33) and (2.34) respectively, the following identity holds true

$$\nabla w \cdot (\nabla w \times u) = 0$$

for all $u$ in $V^{(2)}_k(T)$ or $W^{(2)}_{k-1}(T)$.

**Proof.** Recall that for $u = \{u_0, u_f, u_e\}$ in $V^{(2)}_k(T)$ or $W^{(2)}_{k-1}(T)$, we have $\nabla w \times u = \{\nabla w,0 u, \nabla w,f u\}$. By letting $w = \nabla w \times u$ in (2.17) or (2.34) and using (2.14) and (2.15) or (2.31) and (2.32), we have for any $w$ in $V^{(2)}_k(T)$ or $W^{(2)}_{k+1}(T)$ that

This completes the proof of the lemma.

4. Commutative properties for the WG de Rham complex 1

Denote by $Q^{(1)}_0$, $Q^{(1)}_f$ and $Q^{(1)}_e$ the $L^2$ projection operators onto $P_k(T)$, $P_k(f)$, and $P_k(e)$ respectively. For any $v \in H^1(T) \cap C(T)$, define $Q^{(1)}_h v$ by

$$Q^{(1)}_h v = \{Q^{(1)}_0 v, Q^{(1)}_f v|_f, Q^{(1)}_e v|_e, v|_n\} \in V^{(1)}_k(T).$$
Next, let $Q_{h}^{(2)}$, $Q_{f}^{(2)}$, and $Q_{e}^{(2)}$ be the $L^2$ projection operators onto $V_{k,0}^{(2)}(T)$, $V_{k,f}^{(2)}(f)$, and $V_{k,e}^{(2)}(e)$ respectively. For $u \in H(\text{curl}; T) \cap [C(T)]^3$, define $Q_{h}^{(2)}u$ by

$$Q_{h}^{(2)}u = \{Q_{h}^{(2)}u, Q_{f}^{(2)}(n_f \times (u|_f \times n_f)), Q_{e}^{(2)}(u|_e \cdot t_e)t_e\} \in V_{h}^{(2)}(T).$$

(4.2)

Analogously, with $Q_{h}^{(3)}$ and $Q_{f}^{(3)}$ being the $L^2$ projection operators onto $V_{k,0}^{(3)}(T)$ and $V_{k,f}^{(3)}(f)$ we define

$$Q_{h}^{(3)}w = \{Q_{h}^{(3)}w, Q_{f}^{(3)}(w|_f \cdot n_f)n_f\} \in V_{h}^{(3)}(T)$$

(4.3)

for any $w \in H(\text{div}; T) \cap [C(T)]^3$. Denote by $Q_{h}^{(4)}$ the $L^2$ projection operator from $L^2(T)$ to $V_{h}^{(4)}(T)$.

**Lemma 4.1.** For the projection operators $Q_{h}^{(1)}$ and $Q_{h}^{(2)}$ defined in (4.1) and (4.2) respectively and the weak gradient $\nabla_w$ defined in (2.13), we have

$$\nabla_w(Q_{h}^{(1)}u) = Q_{h}^{(2)}\nabla u \quad \forall u \in H^1(T) \cap C^1(T).$$

(4.4)

**Proof.** By (2.13), one has $\nabla_w Q_{h}^{(1)}u = \{\nabla_w, Q_{h}^{(1)}u, \nabla_w f, Q_{h}^{(1)}u, \nabla_w, Q_{h}^{(1)}u\}$. From (4.2), we have $Q_{h}^{(2)}\nabla u = \{Q_{0}^{(2)}\nabla u, Q_{f}^{(2)}(n_f \times (\nabla u|_f \times n_f)), Q_{e}^{(2)}(\nabla u|_e \cdot t_e)t_e\}$. Thus it suffices to prove

$$\nabla_w, Q_{h}^{(1)}u = Q_{h}^{(2)}\nabla u, \quad \forall f \in F(T),$$

(4.5)

$$\nabla_w f, Q_{h}^{(1)}u = Q_{f}^{(2)}(n_f \times (\nabla u \times n_f)), \quad \forall f \in F(T),$$

(4.6)

$$\nabla_w, Q_{e}^{(1)}u = Q_{e}^{(2)}(\nabla u \cdot t_e)t_e, \quad \forall e \in E(T).$$

(4.7)

To prove (4.5), we have from (2.10) that for $u \in H^1(T) \cap C^1(T)$ and any $\varphi \in V_{h}^{(2)}(T)$,

$$(\nabla_w, Q_{h}^{(1)}u, \varphi) = -(Q_{0}^{(2)} \varphi, \nabla \cdot \varphi) + (Q_{f}^{(1)}u, \varphi \cdot n_f)_{\partial T}$$

$$= -(v, \nabla \cdot \varphi) + (v, \varphi \cdot n_f)_{\partial T}$$

$$= (\nabla v, \varphi)_{\partial T} = (Q_{0}^{(2)} \nabla v, \varphi)_{\partial T},$$

which leads to $\nabla_w, Q_{h}^{(1)}u = Q_{h}^{(2)}\nabla u$ in $T$, and thus proves (4.5).

Next, from (2.11) we have for any $\theta \in V_{k,f}^{(3)}$

$$(\nabla_w, Q_{h}^{(1)}u, \theta \times n_f) = -(Q_{f}^{(1)}u, \nabla \times \theta \cdot n_f) + (Q_{e}^{(1)}u, \theta \cdot t_{\text{ef}})$$

$$= -(v, \nabla \times \theta \cdot n_f) + (v, \theta \cdot t_{\text{ef}})$$

$$= (\nabla v, \theta \times n_f) = (n_f \times (\nabla v \times n_f), \theta \times n_f),$$

(4.6)

which verifies the identity (4.6).

To derive (4.7), from (2.12) we have for $v \in H^1(T) \cap C^1(T)$ and any $\varphi \in V_{k,e}^{(3)}(e)$ that

$$(\nabla_w, Q_{e}^{(1)}u, \varphi t_e) = -(Q_{e}^{(2)} \varphi \cdot t_e) + (v, \varphi t_e \cdot n_{\text{ef}})$$

$$= -(v, \nabla \varphi \cdot t_e) + (v, \varphi t_e \cdot n_{\text{ef}})$$

$$= (\nabla v, \varphi t_e) = (Q_{e}^{(2)} \nabla v \cdot t_e t_e, \varphi t_e),$$

which verifies (4.7). This completes the proof of the lemma. \(\square\)

**Lemma 4.2.** For the projections $Q_{h}^{(2)}$ and $Q_{h}^{(3)}$ defined in (4.2) and (4.3) respectively and the weak curl $\nabla_w \times$ defined in (2.16), we have

$$\nabla_w \times (Q_{h}^{(2)}u) = Q_{h}^{(3)}\nabla \times u \quad \forall u \in H(\text{curl}, T) \cap [C^1(T)]^3.$$
It follows from (2.14) that for \( u \in H(\text{curl}, T) \cap [C^1(T)]^3 \) and any \( \theta \in V_{k_0}^3(T) \),
\[
\left( \nabla_{w,0} \times Q_h^{(2)} u, \theta \right)_T = (Q_h^{(2)} u, \nabla \times \theta)_T + (Q_h^{(2)} (n \times (u \times n)), \theta \times n)_{\partial T}
\]
\[
= (u, \nabla \times \theta)_T + (n \times (u \times n), \theta \times n)_{\partial T}
\]
\[
= (u, \nabla \times \theta)_T + (u, \theta \times n)_{\partial T}
\]
\[
= (\nabla \times u, \theta)_T = (Q_h^{(3)} \nabla \times u, \theta)_T
\]
which verifies (4.9). Next, from (2.15) we have for any \( \tau n_f \in V_{k}^3(f) \),
\[
\left( \nabla_{w,f} \times Q_h^{(2)} u, \tau n_f \right)_f = (Q_h^{(2)} (n_f \times (u \times n_f)), \nabla \times \tau n_f)_f + (Q_h^{(2)} (u \cdot t)_f \cdot \tau t)_f
\]
\[
= (n_f \times (u \times n_f), \nabla \times \tau n_f)_f + ((u \cdot t)_f \cdot \tau t)_f
\]
\[
= (u, \nabla \times \tau n_f)_f + (u, \tau t)_f
\]
\[
= (\nabla \times u \cdot n_f, \tau)_f = (Q_h^{(3)} (\nabla \times u \cdot n_f), \tau n_f)_f,
\]
which leads to (4.10). This completes the proof of the lemma.  

**Lemma 4.3.** For the operator \( R_h^{(3)} \) defined in (4.3), the \( L^2 \) projection operator \( Q_h^{(4)} \) from \( L^2(T) \) to \( V_k^4(T) \), and the weak divergence operator \( \nabla_{w,0} \cdot \) defined in (2.17), the following identity holds true
\[
\nabla_{w} \cdot (Q_h^{(3)} w) = Q_h^{(4)} \nabla \cdot w \quad \forall w \in H(\text{div}; T) \cap [C(T)]^3.
\]

**Proof.** It follows from (2.17) that for \( w \in H(\text{div}, T) \cap [C(T)]^3 \) and any \( \tau \in V_k^4(T) \),
\[
\left( \nabla_{w} \cdot Q_h^{(3)} w, \tau \right)_T = -(Q_h^{(3)} w, \nabla \tau)_T + (Q_h^{(3)} (w \cdot n)_n, \tau n)_{\partial T}
\]
\[
= -(w, \nabla \tau)_T + (w, \tau n)_{\partial T}
\]
\[
= (\nabla \cdot w, \tau)_T = (Q_h^{(4)} \nabla \cdot w, \tau)_T,
\]
which completes the proof of the lemma.  

5. Commutative properties for the WG de Rham complex 2

In this section, we show that the diagram in Fig. 1.2 commutes with properly defined operators \( R_h^{(1)}, R_h^{(2)}, R_h^{(3)} \) and \( R_h^{(4)} \). To this end, denote by \( R_h^{(1)}, R_h^{(2)}, R_h^{(3)} \) the \( L^2 \) projection operators onto \( P_h(T), P_{k-1}(f) \), and \( P_{k-2}(e) \) respectively. For any \( v \in H^1(T) \cap C(T) \), we define \( R_h^{(1)} v \) as follows
\[
R_h^{(1)} v = \{ R_h^{(1)} v, R_h^{(2)} v |_{f}, R_h^{(3)} v |_{e}, v |_{\partial f} \} \in W_h^{(1)}(T).
\]  

Next, denote by \( R_h^{(2)} \) and \( R_h^{(3)} \) the \( L^2 \) projection operators onto \( W_{k-1,0}^2(T), W_{k-2,1}^2(f) \), and \( W_{k-3,2}^2(e) \) respectively. For \( u \in H(\text{curl}; T) \cap [C(T)]^3 \), we define \( R_h^{(2)} u \) as follows
\[
R_h^{(2)} u = \{ R_h^{(2)} u, R_h^{(2)} (n_f \times (u \times n_f)), R_h^{(2)} (u |_{f} \cdot t_e) |_{e} \} \in W_h^{(2)}(T).
\]  

Analogously, with \( R_h^{(3)} \) and \( R_h^{(4)} \) being the \( L^2 \) projection operators onto \( W_{k-2,0}^3(T) \) and \( W_{k-3,1}^3(f) \), we may define \( R_h^{(3)} w \) by
\[
R_h^{(3)} w = \{ R_h^{(3)} w, R_h^{(3)} (w \cdot n_f) |_{f}, R_h^{(3)} (\nabla \times w \cdot n_f) |_{f} \} \in W_h^{(3)}(T)
\]  

for all \( w \in H(\text{div}; T) \cap [C(T)]^3 \). Our fourth operator \( R_h^{(4)} \) is given as the \( L^2 \) projection operator from \( L^2(T) \) to \( W_h^{(4)}(T) \).

**Lemma 5.1.** For the linear operators \( R_h^{(1)} \) and \( R_h^{(2)} \) defined in (5.1) and (5.2) respectively and the weak gradient \( \nabla_{w} \) defined in (2.30), the following identity holds true
\[
\nabla_{w}(R_h^{(1)} v) = R_h^{(2)} \nabla v
\]
for all \( v \in H^1(T) \cap C(T) \).

**Proof.** It follows from (2.30) that \( \nabla_{w} R_h^{(1)} v = \{ \nabla_{w,0} R_h^{(1)} v, \nabla_{w,f} R_h^{(1)} v, \nabla_{w,e} R_h^{(1)} v \} \). By (5.2), we have
\[
R_h^{(2)} \nabla v = \{ R_h^{(2)} \nabla v, R_h^{(2)} (n_f \times (\nabla v \times n_f)), R_h^{(2)} (\nabla v \cdot t_e) |_{e} \}.
\]
Thus we need to prove
\[
\nabla_{w,0} R_h^{(1)} v = R_h^{(2)} \nabla v, \quad \text{in} \ T,
\]
\[
\nabla_{w,f} R_h^{(1)} v = R_h^{(2)} (n_f \times (\nabla v \times n_f)), \quad \text{on} \ f \in F(T),
\]
\[
\nabla_{w,e} R_h^{(1)} v = R_h^{(2)} (\nabla v \cdot t_e) |_{e}, \quad \text{on} \ e \in E(T).
\]  

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From (2.27), we have for any $\varphi \in W^{(2)}_{(k-1,0)}(T)$

$$(\nabla_w, \partial_{n_f}^{(1)} v, \varphi)_T = -(R_0^{(1)} v, \nabla \cdot \varphi)_T + \langle R_0^{(2)} v, \varphi \cdot n \rangle_{\partial T}$$

$$= -(v, \nabla \cdot \varphi)_T + \langle v, \varphi \cdot n \rangle_{\partial T}$$

$$= (\nabla v, \varphi)_T = (R_0^{(2)} \nabla v, \varphi)_T,$$

which proves (5.5).

Next, from (2.28) we have for any $\theta \in W^{(2)}_{(k-2, f)}(T)$

$$((\nabla_w R_0^{(1)} v, \theta \times n_f)_f = -(R_0^{(1)} v, \nabla \times \theta \times n_f)_f + \langle R_0^{(1)} v, \theta \cdot t \rangle_{\partial f}$$

$$= -(v, \nabla \times \theta \times n_f)_f + \langle v, \theta \cdot t \rangle_{\partial f}$$

$$= ([n_f \times (\nabla v \times n_f)]_f, \theta \times n_f)_f$$

which verifies (5.6).

Finally, from (2.29), we have for any $\varphi \in W^{(2)}_{(k-3, e)}(e)$

$$((\nabla_w, \partial_{e_\varphi}^{(1)} v, \varphi t_e)_e = -(R_0^{(1)} v, \nabla \cdot t_e)_e + \langle v, \varphi t_e \cdot n_{3e} \rangle_{\partial e_e}$$

$$= -(v, \nabla \cdot t_e)_e + \langle v, \varphi t_e \cdot n_{3e} \rangle_{\partial e_e}$$

$$= ([n_e \times (\nabla v \times n_e)]_e, \varphi t_e)_e.$$

which proves (5.7). This completes the proof of the lemma. □

**Lemma 5.2.** For the linear operators $R_0^{(2)}$ and $R_0^{(3)}$ defined in (5.2) and (5.3) and the weak curl $\nabla_w \times$ defined in (2.33), the following identity holds true:

$$\nabla_w \times (R_0^{(2)} u) = R_0^{(3)} \nabla \times u \quad \text{for all } u \in H(\text{curl}; T) \cap [C(T)]^3.$$ \hspace{1cm} (5.8)

**Proof.** By (2.33), one has $\nabla_w \times R_0^{(2)} u = \{\nabla_w, 0 \times R_0^{(2)}, u, \nabla_w, f \times R_0^{(2)} u\}. By (5.3)$, we have $R_0^{(3)} \nabla \times u = \{R_0^{(3)} \nabla \times u, R_0^{(3)} (\nabla \times u \cdot n_f) n_f\}$. Thus it suffices to prove

$$\nabla_w, 0 \times R_0^{(2)} u = R_0^{(3)} \nabla \times u, \nabla_w, f \times R_0^{(2)} u = R_0^{(3)} (\nabla \times u \cdot n_f) n_f.$$ \hspace{1cm} (5.9)

First, it follows from (2.31) that for $u \in H(\text{curl}; T) \cap [C(T)]^3$ and any $\theta \in W^{(3)}_{(k-2, 0)}(T)$,

$$((\nabla_w, 0 \times R_0^{(2)} u, \theta) = (R_0^{(2)} u, \nabla \times \theta) + \langle R_0^{(2)} (n \times (u \times n)), \theta \times n \rangle_{\partial T}$$

$$= ([n \times (u \times n)]_T, \theta \times n)_{\partial T}$$

which implies that $\nabla_w, 0 \times R_0^{(2)} u = R_0^{(3)} \nabla \times u$.

Next, from (2.32) we have for any $\tau n \in W^{(k-3, f)}(f)$

$$((\nabla_w, f \times R_0^{(2)} u, \tau n_f)_f = (R_0^{(2)} (n_f \times (u \times n_f)), \nabla \tau \times n_f)_f + \langle R_0^{(2)} (u \cdot t) n_f + \langle u \cdot t \rangle_{\partial f}$$

$$= ([n_f \times (u \times n_f)]_f, \nabla \tau \times n_f)_f + \langle u, \tau \rangle_{\partial f}$$

which implies $\nabla_w, f \times R_0^{(2)} u = R_0^{(3)} (\nabla \times u \cdot n_f) n_f$ on face $f \in F(T)$. This completes the proof of the lemma. □

**Lemma 5.3.** For the linear operator $R_0^{(3)}$ defined in (5.3), the $L^2$ projection $R_0^{(4)}$ from $L^2(T)$ to $W^{(4)}_{(k-3)}(T)$, and the weak divergence $\nabla_w \cdot$ defined in (2.34), the following identity holds true

$$\nabla_w \cdot (R_0^{(3)} w) = R_0^{(4)} \nabla \cdot w \quad \text{for all } w \in H(\text{div}; T) \cap [C(T)]^3.$$ \hspace{1cm} (5.10)
Proof. It follows from (2.34) that for \( w \in H(div, T) \cap [C(T)]^3 \) and any \( \tau \in W_{k-3}^{(4)}(T) \), we have
\[
(\nabla_w \cdot R^{(3)}_{\mathbf{n}} w, \tau)_T = -(R^{(3)}_0 w, \nabla \tau)_T + (R^{(3)}_0 (w \cdot \mathbf{n}), \tau\mathbf{n})_T = (\nabla w, \nabla \tau)_T + (w \cdot \mathbf{n}, \tau)_T = (\nabla \cdot w, \tau)_T = (R^{(4)}_0 \nabla \cdot w, \tau)_T,
\]
which proves the lemma. □

6. Exactness for de Rham complex 1

We show that the weak Galerkin de Rham complex 1 is exact for \( k = 0 \) on tetrahedra and hexahedra.

**Theorem 6.1.** Let \( T \) be a tetrahedron. The following de Rham complex is exact for \( k = 0 \)
\[
\begin{align*}
\mathbb{R}^1 \xrightarrow{I} C^\infty(T) & \xrightarrow{\nabla} [C^\infty(T)]^3 & \xrightarrow{\nabla \times} [C^\infty(T)]^3 & \xrightarrow{\nabla \cdot} C^\infty(T) & \xrightarrow{N} 0 \\
\mathbb{R}^4 \xrightarrow{I_w} V_k^{(1)}(T) & \xrightarrow{\nabla_w} V_k^{(2)}(T) & \xrightarrow{\nabla_w \times} V_k^{(3)}(T) & \xrightarrow{\nabla_w \cdot} V_k^{(4)}(T) & \xrightarrow{N} 0
\end{align*}
\]

(6.1)

Here \( I_w \) is the inclusion map that assigns constant value to \( v_0, v_f, v_e, \) and \( v_n; N \) stands for the null operator.

**Proof.** A necessary condition for exactness is zero difference of dimensions; i.e.,
\[
\sum_{i=0}^{4} (-1)^i \dim V_k^{(i)}(T) = 0,
\]
\[
= 4 \frac{(k+1)(k+2)(k+3)}{6} - 4 \frac{(k+1)(k+2)}{2} - 6(k+1) - 4 + 3 \frac{(k+1)(k+2)(k+3)}{6} + 2 \cdot 4 \frac{(k+1)(k+2)}{2} + 6(k+1) - 3 \frac{(k+1)(k+2)(k+3)}{6} - 4 \frac{(k+1)(k+2)}{2} + 6
\]
\[
= 0,
\]
where we have set \( V_k^{(0)}(T) = \mathbb{R}^4 \).

We claim that the kernel of the operator \( \nabla_w \) has dimension 4. To this end, let \( v \in V_k^{(1)}(T) \in \ker(\nabla_w) \); i.e.
\[
0 = \nabla_w v = [\nabla_w 0, \nabla_w f v, \nabla_w e v].
\]

From the definition (2.12) for \( \nabla_{w,f} v \), we have \( v_n = \alpha_4 \) with a constant \( \alpha_4 \) at all the vertices of \( T \). On each face (triangle) \( f \in F(T) \), one may construct a linear function \( P_{1,f} v \) so that \( P_{1,f} v = v_e \) at the center of each edge \( e \in \partial f \). From the definition (2.11) we have
\[
\nabla_{w,f} v = P_{1,f} v \quad \forall f \in F(T),
\]
where \( \nabla_f \) is the surface gradient operator on the face \( f \). It follows from \( \nabla_{w,f} v = 0 \) that \( \nabla_f P_{1,f} v = v_f \) so that \( P_{1,f} v = \alpha_2 \) for a constant \( \alpha_2 \). This shows that \( v_f = \alpha_2 \) on all edges. Analogously, we may obtain \( v_f = \alpha_2 \) on all faces for a constant \( \alpha_2 \). Finally, \( v_0 = \alpha_1 \) is already a constant that does not enter into the calculation of the weak derivatives for the case of \( k = 0 \). This shows that the dimension of \( \ker(\nabla_w) \) is 4 so that
\[
\dim(\ker(\nabla_w)) = 4.
\]

Since the dimension of \( V_k^{(1)}(T) = 15 \), thus the dimension of \( \ker(\nabla_w) \) is 15 - 4 = 11.

Next, we claim that
\[
\dim(\ker(\nabla_w \times)) = 6. \tag{6.3}
\]

Since \( \dim(V_k^{(4)}) = 1 \) and \( \dim(V_k^{(3)}) = 7 \), we have \( \dim(\ker(\nabla_w \times)) = 1 \) and \( \dim(\ker(\nabla_w \cdot)) = 6 \). It follows that
\[
\ker(\nabla_w \times) = \ker(\nabla_w \cdot)
\]
provided that (6.3) holds true.
From $\dim(V_0^{(2)}) = \dim(\text{Range}(\nabla_w \times)) + \dim(\text{Ker}(\nabla_w \times))$ and (6.3), we have
$$\dim(\text{Ker}(\nabla_w \times)) = \dim(V_0^{(2)}) - 6 = 17 - 6 = 11 = \dim(\text{Range}(\nabla_w)).$$
Hence, we have from (3.1) that
$$\text{Ker}(\nabla_w \times) = \text{Range}(\nabla_w).$$
It remains to prove (6.3). Note that by (3.2), we have $\dim(\text{Range}(\nabla_w \times)) \leq \dim(\text{Ker}(\nabla_w \times)) = 6$. Consider the following orthogonal decomposition of $V_0^{(3)}(T)$:
$$V_0^{(3)}(T) = \text{Range}(\nabla_w \times) \oplus W.$$ 
It is clear that (6.3) is equivalent to $\dim(W) = 1$. For any $w = (w_0, w_f n_f) \in W$, we have
$$(w, \nabla_w \times v)_T = 0 \quad \forall v \in V_0^{(2)},$$
which implies
$$(w_0, \nabla_{w_0} \times v)_T = 0,$$ and on each face $f$:
$$(w_f n_f, \nabla_{w,f} \times v)_f = 0 \quad \forall v \in V_0^{(2)} \times V_{0,f}^{(2)},$$
Using the definition of $w_{0}$ in (2.14) and (6.4), we have
$$\sum_f (w_0, n_f \times v_f)_f = 0 \quad \forall v = (v_0, v_f) \in V_0^{(2)} \times V_{0,f}^{(2)},$$
which implies $w_0 \times n_f = 0$ on each face $f$ so that $w_0 = 0$. Next, it follows from (6.5) that
$$\sum_f (w_f n_f, \nabla_{w,f} \times v)_f = \sum_f ([w_f], v_e)_e = 0,$$
which implies the jump $[w_f]_e = 0$ on all edges. Thus, $w_f$ assumes a constant value at all faces $f \in F(T)$. This shows that the function $w \in W$ have the form $w = (0, c n_f)$ with $c = \text{const}$ so that $\dim(W) = 1$. This verifies the claim (6.3). \(\square\)

**Theorem 6.2.** For $k = 0$, the following de Rham complex is exact on cubic element $T$:
$$\begin{array}{cccc}
\mathbb{R}^1 & I & C^\infty(T) & \nabla & C^\infty(T)^3 & \nabla \times & C^\infty(T)^3 & \nabla \cdot & C^\infty(T) & N & 0 \\
\mathbb{R}^8 & I_T & V_0^{(1)}(T) & \nabla w & V_0^{(2)}(T) & \nabla w \times & V_0^{(3)}(T) & \nabla w \cdot & V_0^{(4)}(T) & N & 0
\end{array}$$

**Proof.** Again, the necessary condition of zero difference of dimensions for exactness holds true, as it can be easily seen that
$$\sum_{i=0}^4 (-1)^{i-1} \dim V_k^{(i)}(T) = -8$$
$$= \frac{(k + 1)(k + 2)(k + 3)}{6} + \frac{6(k + 1)(k + 2)}{2} + 12(k + 1) + 8$$
$$- \frac{3(k + 1)(k + 2)(k + 3)}{6} - 2 \cdot \frac{6(k + 1)(k + 2)}{2} - 12(k + 1)$$
$$+ \frac{3(k + 1)(k + 2)(k + 3)}{6} + \frac{6(k + 1)(k + 2)}{2}$$
$$- \frac{(k + 1)(k + 2)(k + 3)}{6} = 0,$$
where we have set $V_0^{(0)}(T) = \mathbb{R}^8$.

Let us show that the kernel of the operator $\nabla_w$ has dimension 8. In fact, for any $v \in V_0^{(1)}(T) \in \text{Ker}(\nabla_w)$, we have
$$0 = \nabla_w v = \{\nabla_{w,0} v, \nabla_{w,f} v, \nabla_{w,e} v\}.$$
From the definition (2.12) for $\nabla_{w,e} v$, we see that $v_e = \alpha_8$ with a constant $\alpha_8$ at all eight vertices of $T$. On each face (rectangle) $f \in F(T)$, the condition of $\nabla_{w,f} v = 0$ implies $v_e$ has the same value on any parallel edges so that $v_e$ has a
total of 3 independent unknowns. Analogously, the condition of \( \nabla_w \partial_t v = 0 \) implies that \( v_f \) has the same value on any parallel faces so that \( v_f \) has a total of 3 independent unknowns. With the one free unknown for \( v_0 \), we have a total of 8 independent unknowns for functions in \( \text{Ker}(\nabla_w) \) so that

\[
\text{dim}(\text{Ker}(\nabla_w)) = 8, \quad (6.7)
\]

which leads to

\[
\text{Range}(I_w) = \text{Ker}(\nabla_w).
\]

Since the dimension of \( V_0^{(1)}(T) = 27 \), thus the dimension of \( \text{Range}(\nabla_w) = 27 - 8 = 19 \).

Observe that the proof of (6.3) can be adopted without any modification to yield the following result:

\[
\text{dim}(\text{Range}(\nabla_w \times)) = 8. \quad (6.8)
\]

Since \( \text{dim}(V_0^{(4)}) = 1 \) and \( \text{dim}(V_0^{(3)}) = 9 \), we then have \( \text{dim}(\text{Range}(\nabla_w \cdot)) = 1 \) and \( \text{dim}(\text{Ker}(\nabla_w \cdot)) = 8 \). It follows from (6.8) that

\[
\text{Range}(\nabla_w \times) = \text{Ker}(\nabla_w \cdot).
\]

Next, from \( \text{dim}(V_0^{(2)}) = \text{dim}(\text{Range}(\nabla_w \times)) + \text{dim}(\text{Ker}(\nabla_w \times)) \) and (6.8), we have

\[
\text{dim}(\text{Ker}(\nabla_w \times)) = \text{dim}(V_0^{(2)}) - 8 = 27 - 8 = 19 = \text{dim}(\text{Range}(\nabla_w)).
\]

Hence, we have from (3.1) that

\[
\text{Ker}(\nabla_w \times) = \text{Range}(\nabla_w). \quad \square
\]

References