CONSTRUCTING ORDER TWO SUPERCONVERGENT DISCONTINUOUS FINITE ELEMENTS ON RECTANGULAR MESH. PART 2. THE CDG METHOD

XIU YE AND SHANGYOU ZHANG

ABSTRACT. A novel conforming discontinuous Galerkin (CDG) finite element method is introduced for Poisson equation on rectangular mesh. This CDG method with discontinuous $P_k$ elements converges two orders faster than the continuous finite element counterpart. Superconvergence of order two for the CDG finite element solution is proved in an energy norm and the $L^2$ norm. A local post-process is defined which lifts a $P_k$ CDG solution to a discontinuous $P_{k+2}$ solution. It is proved that the lifted $P_{k+2}$ solution converges at the optimal order. The numerical test confirms the theoretic findings.

1. INTRODUCTION

A new conforming discontinuous Galerkin finite element method is developed for the second order elliptic problem in two dimension:

\begin{align}
-\Delta u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align}

where $\Omega$ can be subdivided into rectangular meshes.

Some discontinuous finite element methods use discontinuous $P_k$ elements ($P_k$ denotes the $k$th degree polynomial set) such as the CDG method in [5] and the interior penalty discontinuous Galerkin (IPDG) method in [1], while their continuous counterpart conforming finite element method uses continuous $P_k$ element. Since discontinuous $P_k$ polynomial introduces many more degrees of freedom, one would expect higher order convergence for the finite element methods using discontinuous $P_k$ elements. Unfortunately, so far all IPDG and CDG methods have the same optimal convergence rate as their continuous counterpart. Can we develop a finite element method with discontinuous $P_k$ element, that utilizes appropriately all the additional unknowns introduced by the discontinuity of $k$th degree polynomial to achieve higher order convergence than its continuous counterpart? Finding an answer to this question is far from trivial. We give an affirmative answer to this 50-year old question, which was raised when the first discontinuous finite element method was proposed.

We have given a positive answer to the question by developing a CDG finite element method in one dimension [9]. Superconvergence of order two for the CDG finite element solution is obtained in 1D [9]. The objective of this paper is to

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introduce a finite element method with discontinuous $P_k$ element that converges to the true solution with order two superconvergence in two dimensions.

The CDG method gets its name by combining the features of both conforming finite element method and discontinuous Galerkin (DG) finite element method. It has the flexibility of using discontinuous approximation and simplicity in formulation of the conforming finite element method. Let us use the model problem (1.1) as an example to explain what it means. The conforming finite element method with continuous $P_k$ element has the formulation: find $u_h \in V_h \subset H^1_0(\Omega)$ such that

$$\sum_{T \in \mathcal{T}_h} (\nabla u_h, \nabla v)_{T} - \sum_{e \in \mathcal{E}_h} \left( \{\nabla u_h\}[v_e] + \{\nabla v\}[u_h] - \alpha h_e^{-1}[u_h][v_e]\right)$$

$$= (f, v) \quad \forall v \in V_h \subset L^2(\Omega),$$

where $\alpha$ is called penalty parameter that needs to be tuned. When discontinuous approximation is used, finite element formulations tend to be more complex than (1.3) to ensure connection of discontinuous function across element boundary. The CDG method shares the same finite element space as the IPDG method in (1.4) but it has a simple formulation similar to (1.3): find $u_h \in V_h \subset L^2(\Omega)$ such that

$$\nabla w u_h \cdot \nabla w v = (f, v) \quad \forall v \in V_h \subset L^2(\Omega),$$

where $\nabla w$ is the so called weak gradient specially designed for discontinuous $P_k$ polynomials. The reason of the CDG method maintaining its simple formulation is the use of the concept of weak derivatives, introduced in the weak Galerkin (WG) method. The weak Galerkin finite element methods were first proposed and analyzed in [8]. The CDG methods have been developed for diffusion problem [4, 5, 6], and for biharmonic equations [7].

In this paper, we develop a new CDG method on rectangular mesh, which fully utilizes all the degrees of freedom of the discontinuous $P_k$ polynomial to obtain two orders higher convergence rate than its continuous counterpart. This paper is the Part two of the sequence. In Part one [10], we have proved that the corresponding WG method converges to the true solution with order two superconvergence with the help of additional variable $v_b$ defined on element boundary. Contrary to the IPDG and CDG methods, both the weak Galerkin method and the Hybrid high-order methods [3] use discontinuous $P_k$ element plus additional variables associated with element boundary. It is much more difficult to develop a CDG method with order two superconvergence because it has many fewer unknowns. Further, the $P_k$ CDG solution is lifted to a $P_{k+2}$ solution elementwise, which converges at the optimal order. Order two superconvergence of the CDG solution is confirmed theoretically and numerically.

2. CDG Finite Element Scheme

Let $\mathcal{T}_h$ be a partition of the domain $\Omega$ consisting of rectangles. Denote by $\mathcal{E}_h$ the set of all edges in $\mathcal{T}_h$, and by $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial \Omega$ the set of all interior edges. For every element $T \in \mathcal{T}_h$, we denote by $h_T$ its diameter and by $h = \max_{T \in \mathcal{T}_h} h_T$ for $\mathcal{T}_h$. 
For a given integer \( k \geq 1 \), let \( V_h \) be the CDG finite element space associated with \( \mathcal{T}_h \) by
\[
(2.1) \quad V_h = \{ v \in L^2(\Omega) : v|_T \in P_k(T), \ T \in \mathcal{T}_h \}.
\]

For the purpose of error analysis, we also define a weak Galerkin (WG) finite element space as follows
\[
(2.2) \quad \hat{V}_h = \{ v = \{ v_0, v_b \} : v_0|_T \in P_k(T), v_b|_e \in P_{k+1}(e), \ e \subseteq \partial T, \ T \in \mathcal{T}_h, \ v_b|_{\partial \Omega} = 0 \}.
\]

We would like to emphasize that any function \( v \in \hat{V}_h \) has a single value \( v_b \) on each edge \( e \in \mathcal{E}_h \).

For \( T \in \mathcal{T}_h \), \( \text{BDM}_{k+1}(T) = [P_{k+1}(T)]^2 \otimes \text{curl}(x^{k+2}y) \oplus \text{curl}(xy^{k+2}) \) in [2]. For \( v = \{ v_0, v_b \} \in \hat{V}_h \), a weak gradient \( \nabla_w v \) is a piecewise vector valued polynomial such that on each \( T \in \mathcal{T}_h \), \( \nabla_w v|_T \in \text{BDM}_{k+1}(T) \) satisfies
\[
(2.3) \quad (\nabla_w v, q)_T = (\nabla v_0, q)_T + (v_0 - v_b, q \cdot n)_{\partial T} \quad \forall q \in \text{BDM}_{k+1}(T).
\]

For a given \( e \in \mathcal{E}_h \), let \( U_e \) denote the union of four rectangles with \( e \) between two of them, as shown in Figure 3.1. For given \( v \in V_h \) and \( e \in \mathcal{E}_h \), define \( E_{k+2} : \prod_{T \in U_e} P_k(T) \to P_{k+2}(U_e) \) as a lifting operator such that \( E_{k+2} v \) is the least-squares solution of
\[
(2.4) \quad (E_{k+2} v, q)_{U_e} = (v, q)_{U_e} \quad \forall q \in \prod_{T \in U_e} P_k(T).
\]

In general, \( \Pi_k \) also denotes a generic local \( L^2 \) projection onto \( P_k(D) \) where the set \( D \) could be a rectangle \( T \in \mathcal{T}_h \) or an edge \( e \in \mathcal{E}_h \). We define an embedding operator \( \Pi_k : V_h \to \hat{V}_h \) such that for \( v \in V_h \)
\[
(2.5) \quad E_h v = \{ v, v_b \} \in \hat{V}_h,
\]

where
\[
(2.6) \quad v_b|_e = \begin{cases} 0, & \text{if } e \subseteq \partial \Omega, \\ \Pi_{k+1}(E_{k+2} v), & \text{if } e \in \mathcal{E}_h^0. \end{cases}
\]

Since \( E_h v \in \hat{V}_h \), \( \nabla_w E_h v \) can be calculated by (2.3). For \( v \in V_h \), its weak gradient \( \nabla_w v \) is defined as
\[
(2.7) \quad \nabla_w v = \nabla_w E_h v.
\]

The CDG finite element method is to find \( u_h \in V_h \) such that
\[
(2.8) \quad (\nabla_w u_h, \nabla_w v) = (f, v) \quad \forall v \in V_h.
\]

3. Approximation property of the operator \( E_{k+2} \)

**Lemma 3.1.** For \( k \geq 1 \), the pseudo-projection \( E_{k+2} \Pi_k : H^{k+3}(U_e) \to P_{k+2}(U_e) \) has an order of \( k+3 \) accuracy. That is,
\[
(3.1) \quad \| E_{k+2} \Pi_k u - u \|_{0, U_e} \leq C h^{k+3} |u|_{k+3, U_e}.
\]

**Proof.** Let \( e = [-1, 0] \times \{0\} \) be an horizontal edge and \( U_e = T_1 \cup T_2 \cup T_3 \cup T_4 \) be the 4 squares used for defining \( E_{k+2} \), shown in Figure 3.1. Here we did a tensor 2-affine mapping which shifts and scales the four rectangles to \( T_1 = [-1, 0] \times [-1, 0] \), \( T_2 = [0, 1] \times [-1, 0] \), \( T_3 = [0, 1] \times [0, 1] \), and \( T_4 = [-1, 0] \times [0, 1] \).
Figure 3.1. $U_e = [-1, 1] \times [-1, 1]$ for a horizontal edge $e$.

We show next the local $L_2$ projection $\Pi_k$ is an injection mapping when restricted on $P_{k+2}(U_e)$. That is, if $\Pi_k p_{k+2} = 0$ for some $p_{k+2} \in P_{k+2}(U_e)$, then $p_{k+2} = 0$. We expand this $P_{k+2}(U_e)$ polynomial under the product Legendre basis,

$$p_{k+2}(x, y) = \sum_{0 \leq i, j \leq k+2} c_{ij} L_i(x) L_j(y), \quad (3.2)$$

where $L_i(x)$ is the $i$-th Legendre polynomial on $[-1, 1]$ which is orthonormal to all Legendre polynomials. We will show that all the coefficients of $p_{k+2}$ below are zero:

$$c_{k+2,0} \quad c_{k+1,0} \quad c_{k+1,1} \quad c_{k,0} \quad c_{k,1} \quad c_{k,2} \quad \vdots \quad \vdots \quad \vdots \quad c_{0,k+1} \quad c_{0,k+2} \quad c_{0,0} \quad c_{0,1} \quad \cdots \quad \cdots \quad \cdots \quad c_{0,k+1} \quad c_{0,k+2}$$

We test (3.2) by two-piece polynomials $p(x)L_0(y)$ where $p(x) \in P_k(-1, 0) \times P_k(0, 1)$,

$$0 = (p_{k+2}, p(x)L_0(y))_{U_e} = \int_{-1}^{1} \sum_{i=0}^{k+2} c_{i,0} L_i(x)p(x)dx$$

$$= \int_{-1}^{1} q_{k+2}(x)p(x)dx.$$

That is, the 1D $P_{k+2}$ polynomial $q_{k+2}$ is orthogonal to all two-piece $P_k$ polynomials on $(-1, 0) \cup (0, 1)$. By Lemma 3.1 of [9], $q_{k+2}(x) = 0$, if $k \geq 1$. Thus all the first-column coefficients $c_{i,0}$ of $p_{k+2}$ in (3.2) are zero.

When the first-column terms are gone, we test (3.2) by another two-piece polynomial $p(x)L_1(y)$ where $p(x) \in P_{k-1}(-1, 0) \times P_{k-1}(0, 1)$,

$$0 = (p_{k+2}, p(x)L_1(y))_{U_e} = \int_{-1}^{1} \sum_{i=0}^{k+1} c_{i,1} L_i(x)p(x)dx$$

$$= \int_{-1}^{1} q_{k+1}(x)p(x)dx.$$

Again, by Lemma 3.1 of [9], $q_{k+1}(x) = 0$, if $k-1 \geq 1$. Thus all the second-column coefficients $c_{i,1}$ of $p_{k+2}$ in (3.2) are also zero. We repeat the above selection until
the third column from right. For this column’s coefficients, if we continue testing (3.2) by a two-piece polynomial \( p(x)L_k(y) \) where \( p(x) \in P_0(-1,0) \times P_0(0,1) \), we would get

\[
0 = (p_{k+2}, p(x)L_k(y))_{U_e} = \int_{-1}^{1} \sum_{i=0}^{2} c_{i,k} L_i(x)p(x)dx
= \int_{-1}^{1} q_2(x)p(x)dx.
\]

However, this time the two-\( P_0 \) polynomials \( p(x) \) cannot enforce \( q_2(x) = 0 \), as a condition \( k \geq 1 \) is required in Lemma 3.1 of [9].

To force the last three columns of coefficients in (3.2) to zero, we select testing functions for the bottom three rows, \( L_0(x)p_k(y), L_1(x)p_{k-1}(y), \) and \( L_2(x)p_{k-2}(y) \). For the same reason as above, we obtain zero for all coefficients of last three rows of \( p_{k+2} \), if \( k \geq 3 \).

If \( k = 2 \), the coefficients of both third row and third column are not determinable by the above selections. When \( k = 1 \), the coefficients of both third row and third column from right cannot be forced to zero by the above selections. We deal with these two cases specially.

When \( k = 1 \), the polynomial \( p_{k+2} \) in (3.2), after above testings, has the following expansion,

\[
p_{k+2}(x,y) = c_{10} L_1(x)L_0(y) + c_{11} L_1(x)L_1(y) + c_{12} L_1(x)L_2(y)
+ c_{01} L_0(x)L_1(y) + c_{21} L_2(x)L_1(y).
\]

We note that only \( c_{11} \) is unknown in (3.3) as the other \( c_{ij} \) are determined to be zero from the testings on row 1, row 2, column 1 from bottom, and column 2 from bottom. But we keep them here and do two uniform testings. This time we cannot use one-piece Legendre polynomials and we have to use four-piece testing polynomials instead of two-piece polynomials used above. Let \( 0 \neq q_0(x) \in P_0(-1,0) \times P_0(0,1) \) be a two-piece \( P_0 \) polynomial

\[
q_0(x) = \begin{cases} -1 & x \in (-1,0) \\ 1 & x \in (0,1), \end{cases}
\]

such that

\[
\int_{-1}^{1} q_0(x)L_i(x)dx = \begin{cases} 0, & i = 0, 2, \\ c \neq 0 & i = 1. \end{cases}
\]

Testing (3.3) by \( q_0(x)p(y) \) with \( p(y) \in P_1(-1,0) \times P_1(0,1) \), we get

\[
0 = (p_{k+2}, q_0(x)p(y))_{U_e} = c \int_{-1}^{1} \sum_{i=0}^{2} c_{i,1} L_i(y)p(y)dy
= c \int_{-1}^{1} q_2(y)p(y)dy,
\]

where \( q_2(y) \in P_2(U_e) \). As \( p(y) \) is a two-piece \( P_1 \) polynomial, by [9], it would force all \( P_3 \) polynomials (here \( q_2 \in P_2 \subset P_3 \)) to zero. Thus \( c_{1i} = 0, i = 0, 1, 2 \). We test (3.3) by \( p(x)q_0(y) \), in the other direction. Then \( c_{1i} = 0, i = 0, 1, 2 \). Therefore \( p_{k+2}(x) = 0 \) if \( \Pi_k p_{k+2}(x) = 0 \), for \( k = 1 \).
When \( k = 2 \), the polynomial \( p_{k+2} \) in (3.2), after the earlier testings, has the following non-zero terms left,

\[
p_{k+2}(x, y) = c_{20} L_2(x) L_0(y) + c_{21} L_2(x) L_1(y) + c_{22} L_2(x) L_2(y) \\
+ c_{02} L_0(x) L_2(y) + c_{12} L_1(x) L_2(y)
\]

\( = c_{22} L_2(x) L_2(y) \).

Testing (3.4) by \( q_{1/2}(x) q_{1/2}(y) \), where

\[
q_{1/2}(x) = \begin{cases} 
-1 & x \in (-1, 0) \\
0 & x \in (0, 1), 
\end{cases}
\]

we get

\[
0 = (p_{k+2}, q_{1/2}(x) q_{1/2}(y))_{U_e} = c^2 c_{2,2}.
\]

Here \( c = \int_{-1}^{1} -L_2(x) dx > 0 \), and \( c_{2,2} = 0 \). Therefore \( p_{k+2}(x) = 0 \) if \( \Pi_k p_{k+2}(x) = 0 \), for \( k = 2 \).

By the definition of \( E_{k+2} \) in (2.4), \( E_{k+2} \) is the pseudo-inverse of \( \Pi_k |_{P_{k+2}(U_e)} \).

Since we proved above \( \Pi_k \) is injective on \( P_{k+2}(U_e) \), the pseudo-inverse would be itself,

\[
(3.5) \quad E_{k+2} \Pi_k p = p \quad \forall p \in P_{k+2}(U_e).
\]

(3.1) is proved by the following triangle inequality and (3.5),

\[
\| E_{k+2} \Pi_k u - u \|_{0, U_e} \leq \| E_{k+2} \Pi_k u - \Pi_{k+2, U_e} u \|_{0, U_e} + \| \Pi_{k+2, U_e} u - u \|_{0, U_e}
\]

\[
= \| E_{k+2} \Pi_k (u - \Pi_{k+2, U_e} u) \|_{0, U_e} + \| \Pi_{k+2, U_e} u - u \|_{0, U_e}
\]

\[
\leq C \| u - \Pi_{k+2, U_e} u \|_{0, U_e} + \| \Pi_{k+2, U_e} u - u \|_{0, U_e}
\]

\[
\leq C \| u \|_{0, U_e} + \| \Pi_{k+2, U_e} u - u \|_{0, U_e}
\]

\[
\leq C h^{k+3} | u |_{k+3, U_e},
\]

where we used the fact \( E_{k+2} \) is a bounded operator by its definition (2.4). \( \square \)

4. The related WG finite element method

We will use the results established in Part 1 of the sequence paper [10] for the corresponding WG method to analyze the CDG finite element solution from (2.8).

The weak Galerkin finite element method in [10] is to find \( \tilde{u}_h = \{ \tilde{u}_0, \tilde{u}_b \} \in \tilde{V}_h \) such that

\[
(\nabla w \tilde{u}_h, \nabla w v) = (f, v_0) \quad \forall v = \{ v_0, v_b \} \in \tilde{V}_h.
\]

Lemma 4.1. [10] The following two norms are equivalent,

\[
C_1 \| v \|_{1, h} \leq \| \nabla w v \|_0 \leq C_2 \| v \|_{1, h} \quad \forall v = \{ v_0, v_b \} \in \tilde{V}_h,
\]

where

\[
\| v \|_{1, h}^2 = \sum_{T \in T_h} (\| \nabla v_0 \|_{0, T}^2 + h_T^{-1} \| v_0 - v_b \|_{0, T}^2).
\]

Lemma 4.2. The CDG finite element method (2.8) has a unique solution.
Proof. Let \( \varepsilon_h = u_h^1 - u_h^2 \) where both \( u_h^1 \) and \( u_h^2 \) are the CDG solution of (2.8). Then we have
\[
(\nabla_w \varepsilon_h, \nabla_w v) = 0 \quad \forall v \in V_h.
\]
Letting \( v = \varepsilon_h \) in (4.3) gives
\[
\|\nabla_w \varepsilon_h\|_0 = 0.
\]
Using the equation above, (4.2) and (2.7), we have
\[
\|E_h \varepsilon_h\|_{1,h} \leq C\|\nabla_w E_h \varepsilon_h\|_0 = C\|\nabla_w \varepsilon_h\|_0 = 0.
\]
By the definition of \( \| \cdot \|_{1,h} \), we have \( \varepsilon_h = 0 \) and complete the proof of the lemma. \( \Box \)

Lemma 4.3. [10] Let \( \phi \in H^1(\Omega) \). Then on any \( T \in T_h \), we have
\[
\nabla_w (Q_h \phi) = \Pi_{k+1}(\nabla \phi),
\]
where \( Q_h : H_h^0(\Omega) \to \hat{V}_h \) is defined by \( Q_h u = \{\Pi_k u, \Pi_{k+1} u\} \).

Theorem 4.1. [10] Let \( u \in H^{k+3}(\Omega) \cap H_0^1(\Omega) \) be the exact solution of (1.1). Let \( \tilde{u}_h = \{\tilde{u}_0, \tilde{u}_b\} \in V_h \) be the WG finite element solution of (4.1). Then
\[
\|\nabla_w (Q_h u - \tilde{u}_h)\|_0 \leq C h^{k+2}|u|_{k+3},
\]
\[
\|\Pi_k u - \tilde{u}_0\|_0 \leq C h^{k+3}|u|_{k+3}.
\]

5. Superconvergence in energy norm

We start this section by deriving the following equation, which plays an important role in the superconvergence analysis of the CDG method. Subtracting (2.8) from (4.1) implies
\[
(\nabla_w (\tilde{u}_h - u_h), \nabla_w v) = 0 \quad \forall v \in V_h.
\]

Lemma 5.1. Let \( u \in H^{k+3}(\Omega) \). Then we have
\[
\|\nabla_w (Q_h u - \Pi_k u)\|_0 \leq C h^{k+2}|u|_{k+3}.
\]
Proof. Recall \( Q_h u = \{\Pi_k u, \Pi_{k+1} u\} \) and \( E_k \Pi_k u = \{\Pi_k u, \Pi_{k+1} E_{k+2} \Pi_k u\} \). Letting \( q = \nabla_w (Q_h u - \Pi_k u) \) in (2.3) and using the trace, inverse inequality and (3.1) yield
\[
\|\nabla_w (Q_h u - \Pi_k u)\|_0^2 = \|\nabla_w (Q_h u - E_k \Pi_k u)\|_0^2
\]
\[
= \sum_{T \in T_h} \langle \Pi_{k+1} u - \Pi_{k+1} E_k \Pi_{k+2} \Pi_k u, q \rangle_{\partial T}
\]
\[
= \sum_{T \in T_h} \langle u - E_{k+2} \Pi_k u, q \rangle_{\partial T}
\]
\[
\leq \left( \sum_{T \in T_h} h_T^{-1} \|u - E_{k+2} \Pi_k u\|_{0,\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_h} h_T \|q\|_{0,\partial T}^2 \right)^{1/2}
\]
\[
\leq C \left( \sum_{T \in T_h} h_T^{-2} \|u - E_{k+2} \Pi_k u\|_{0,\partial T}^2 + \|\nabla(u - E_{k+2} \Pi_k u)\|_{0,\partial T} \right)^{1/2} \|q\|_0
\]
\[
\leq C h^{k+2} |u|_{k+3} \|\nabla_w (Q_h u - \Pi_k u)\|_0.
\]
We complete the proof of the lemma. \( \Box \)
Lemma 5.2. Let u ∈ $H^{k+3}(\Omega)$. Then we have

$$\|\nabla w(\tilde{u}_h - u_h)\|_0 \leq Ch^{k+2}|u|_{k+3}. \quad (5.4)$$

Proof. It follows from (5.1),

$$\|\nabla w(\tilde{u}_h - u_h)\|_0^2 = (\nabla w(\tilde{u}_h - u_h), \nabla w(\tilde{u}_h - u_h))$$
$$= (\nabla w(\tilde{u}_h - u_h), \nabla w(\tilde{u}_h - \Pi_k u))$$
$$\leq \|\nabla w(\tilde{u}_h - u_h)\|_0 \|\nabla w(\tilde{u}_h - \Pi_k u)\|_0,$$

which implies

$$\|\nabla w(\tilde{u}_h - u_h)\|_0 \leq \|\nabla w(\tilde{u}_h - \Pi_k u)\|_0. \quad (5.5)$$

Applying the estimates (4.6) and (5.2) to (5.6), we obtain

$$\|\nabla w(\tilde{u}_h - u_h)\|_0 \leq \|\nabla w(\tilde{u}_h - \Pi_k u)\|_0 \leq C h^{k+2}|u|_{k+3}. \quad (5.6)$$

The proof is completed. \hfill \Box

The order two superconvergence of the CDG solution in an energy norm is obtained in the following theorem.

Theorem 5.1. Let u ∈ $H^{k+3}(\Omega) \cap H^1_0(\Omega)$ be the exact solution of (1.1). Let $u_h \in V_h$ be the CDG solution of (2.8). Then

$$\|\nabla w(\Pi_k u - u_h)\|_0 \leq C h^{k+2}|u|_{k+3}. \quad (5.8)$$

Proof. By (4.6), (5.4) and (5.2), we have

$$\|\nabla w(\Pi_k u - u_h)\|_0$$
$$\leq \|\nabla w(\Pi_k u - Q_h u)\|_0 + \|\nabla w(Q_h u - \tilde{u}_h)\|_0 + \|\nabla w(\tilde{u}_h - u_h)\|_0$$
$$\leq C h^{k+2}|u|_{k+3},$$

which finishes the proof of the lemma. \hfill \Box

6. Superconvergence in L2 norm

We use the duality argument to obtain superconvergence of the CDG finite element solution in the $L^2$ norm. The corresponding dual problem seeks $w \in H^1_0(\Omega)$ satisfying

$$-\Delta w = \tilde{u}_0 - u_h \text{ in } \Omega. \quad (6.1)$$

Recall that $\tilde{u}_h = \{\tilde{u}_0, \tilde{u}_b\}$ and $u_h$ are the solutions of the WG method (4.1) and the CDG method (2.8) respectively. Assume that the following $H^2$-regularity holds

$$\|w\|_2 \leq C\|\tilde{u}_0 - u_h\|_0. \quad (6.2)$$

In the next theorem, we will prove the order two superconvergence of the CDG solution in the $L^2$-norm.

Theorem 6.1. Let u ∈ $H^{k+3}(\Omega) \cap H^1_0(\Omega)$ be the exact solution of (1.1). Let $u_h \in V_h$ be the CDG solution of (2.8). Then

$$\|\Pi_k u - u_h\|_0 \leq C h^{k+3}|u|_{k+3}. \quad (6.3)$$
Proof. Let \( \hat{w}_h \in \hat{V}_h \) be the weak Galerkin finite element solution of (6.1) defined in (4.1), i.e.,

\[
(\nabla_w \hat{w}_h, \nabla_w v) = (\hat{u}_0 - u_h, v_0) \quad \forall v = \{v_0, v_h\} \in \hat{V}_h.
\]

Letting \( v = \hat{u}_h - E_h u_h \in \hat{V}_h \) in (6.4) and by (5.1) and (2.7), we have

\[
\|\hat{u}_0 - u_h\|^2_0 = (\nabla_w \hat{w}_h, \nabla_w (\hat{u}_h - u_h))
= (\nabla_w (\hat{w}_h - \Pi_k w), \nabla_w (\hat{u}_h - u_h))
\leq \|\nabla_w (\hat{w}_h - \Pi_k w)\|_0 \|\nabla_w (\hat{u}_h - u_h)\|_0.
\]

It follows from the estimates (4.6) and (5.2),

\[
\|\nabla_w (\hat{w}_h - \Pi_k w)\|_0 \leq \|\nabla_w (\hat{w}_h - Q_h w)\|_0 + \|\nabla_w (Q_h w - \Pi_k w)\|_0
\leq C_h\|w\|_2.
\]

Using (6.6) and (5.4), (6.5) becomes,

\[
\|\hat{u}_0 - u_h\|^2_0 \leq \|\nabla_w (\hat{w}_h - \Pi_k w)\|_0 \|\nabla_w (\hat{u}_h - u_h)\|_0
\leq C_h\|w\|_2 \|\nabla_w (\hat{u}_h - u_h)\|_0
\leq C_h^{k+3}\|u|_{k+3}\|w\|_2.
\]

It follows from (6.7) and (6.2),

\[
\|\hat{u}_0 - u_h\|_0 \leq C_h^{k+3}\|u|_{k+3}.
\]

Using (4.7) and (6.8), we have

\[
\|\Pi_k u - u_h\|_0 \leq \|\Pi_k u - \hat{u}_0\|_0 + \|\hat{u}_0 - u_h\|_0
\leq C_h^{k+3}\|u|_{k+3}.
\]

We have proved the theorem. \(\square\)

7. A locally lifted \( P_{k+2} \) solution

In last section, we proved that the \( P_k \) conforming discontinuous Galerkin solution is two-order superconvergent, i.e., it converges at order \( k + 3 \) in \( L^2 \) norm. We next define a local post-process lifting such a \( P_k \) solution to an optimal-order \( P_{k+2} \) solution.

On each element \( T \), we compute a solution \( \hat{u}_h \in \Pi_{T \in \mathcal{T}_h} P_{k+2}(T) \) by

\[
(\nabla \hat{u}_h - \nabla w u_h, \nabla v)_{T} = 0 \quad \forall v \in P_{k+2}(T) \setminus P_0(T),
\]

\[
(\hat{u}_h - u_{0}, v)_{T} = 0 \quad \forall v \in P_0(T).
\]

To show the uniqueness of the above square linear system of equations (7.1)–(7.2), letting \( u_h = 0 \), we get from (7.1) that \( \|\nabla \hat{u}_h\|_0^2 = 0 \) and \( \hat{u}_h \) is a constant on each \( T \). By (7.2), the constant is zero. As the linear system is square and finite dimensional, the uniqueness implies the existence of solution.

**Theorem 7.1.** Let \( u \in H^1_0(\Omega) \cap H^{k+3}(\Omega) \) be the exact solution of (1.1)–(1.2). Let \( \hat{u}_h \in \Pi_{T \in \mathcal{T}_h} P_{k+2}(T) \) be the locally lifted solution of (7.1)–(7.2). Then there exists a constant \( C \) such that

\[
\|u - \hat{u}_h\|_0 \leq C h^{k+3}\|u|_{k+3}.
\]

**Proof.** It is same as the proof of Theorem 6.1 in the part one of this paper [10]. \(\square\)
8. Numerical Experiments

We solve problem (1.1) with $\Omega = (0,1)^2$. We choose $f$ and the boundary value $g$ so that the exact solution is

\begin{equation}
(8.1) 
    u(x,y) = \sin \pi x \sin \pi y.
\end{equation}

We take the domain square as the first mesh, and subdivide each square into four to get subsequent meshes, as shown in Figure 8.1. The results of $P_1$, $P_2$, $P_3$ and $P_4$ CDG methods are listed in Table 8.1. Two orders of superconvergence are obtained for each element, in both $L^2$ and $H^1$-like norms.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{grid.png}
\caption{The first three levels of square grids used in the computation.}
\end{figure}

As we have order two superconvergence, we lift each $P_k$ CDG finite element solution $u_h$ to a $P_{k+2}$ solution $\hat{u}_h$ elementwise. From Table 8.2, the lifted $P_{k+2}$ solution converges at order $k+3$ in $L^2$ norm, two orders than that of the original $P_k$ CDG solution.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Grid & $\|Q_h u - u_h\|_0$ & Rate & $\|Q_h u - u_h\|$ & Rate \\
\hline
\hline
& & & & \\
\hline
The $P_1$ CDG finite element & & & & \\
$6$ & $0.786\times10^{-6}$ & $4.01$ & $0.165\times10^{-3}$ & $3.00$ \\
$7$ & $0.490\times10^{-7}$ & $4.01$ & $0.205\times10^{-4}$ & $3.00$ \\
$8$ & $0.305\times10^{-8}$ & $4.00$ & $0.256\times10^{-5}$ & $3.00$ \\
& & & & \\
\hline
The $P_2$ CDG finite element & & & & \\
$5$ & $0.863\times10^{-6}$ & $4.89$ & $0.158\times10^{-3}$ & $3.92$ \\
$6$ & $0.278\times10^{-7}$ & $4.96$ & $0.101\times10^{-4}$ & $3.97$ \\
$7$ & $0.880\times10^{-9}$ & $4.98$ & $0.638\times10^{-6}$ & $3.99$ \\
& & & & \\
\hline
The $P_3$ CDG finite element & & & & \\
$4$ & $0.165\times10^{-5}$ & $6.03$ & $0.212\times10^{-3}$ & $5.02$ \\
$5$ & $0.255\times10^{-7}$ & $6.01$ & $0.661\times10^{-5}$ & $5.00$ \\
$6$ & $0.398\times10^{-9}$ & $6.00$ & $0.207\times10^{-6}$ & $5.00$ \\
& & & & \\
\hline
The $P_4$ CDG finite element & & & & \\
$3$ & $0.167\times10^{-4}$ & $6.89$ & $0.138\times10^{-2}$ & $5.94$ \\
$4$ & $0.132\times10^{-6}$ & $6.98$ & $0.217\times10^{-4}$ & $5.99$ \\
$5$ & $0.105\times10^{-8}$ & $6.98$ & $0.340\times10^{-6}$ & $6.00$ \\
\hline
\end{tabular}
\caption{The error and the convergence rate for problem (8.1).}
\end{table}
Table 8.2. The errors of $P_k$ WG solution $u_h$ and lifted $P_{k+2}$ solution $\tilde{u}_h$, and the convergence rate for problem (8.1).

<table>
<thead>
<tr>
<th>Grid</th>
<th>$|u - u_h|_0$ Rate</th>
<th>$|u - \tilde{u}_h|_0$ Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$ CDG solution</td>
<td>Lifted $P_3$ solution</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.175E-02 2.00</td>
<td>0.110E-05 4.00</td>
</tr>
<tr>
<td>7</td>
<td>0.438E-03 2.00</td>
<td>0.687E-07 4.00</td>
</tr>
<tr>
<td>8</td>
<td>0.109E-03 2.00</td>
<td>0.430E-08 4.00</td>
</tr>
<tr>
<td>$P_2$ CDG solution</td>
<td>Lifted $P_4$ solution</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.271E-03 2.99</td>
<td>0.944E-06 4.91</td>
</tr>
<tr>
<td>6</td>
<td>0.339E-04 3.00</td>
<td>0.302E-07 4.96</td>
</tr>
<tr>
<td>7</td>
<td>0.424E-05 3.00</td>
<td>0.955E-09 4.98</td>
</tr>
<tr>
<td>$P_3$ CDG solution</td>
<td>Lifted $P_5$ solution</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.237E-03 3.99</td>
<td>0.181E-05 6.02</td>
</tr>
<tr>
<td>5</td>
<td>0.149E-04 4.00</td>
<td>0.283E-07 6.00</td>
</tr>
<tr>
<td>6</td>
<td>0.930E-06 4.00</td>
<td>0.443E-09 5.99</td>
</tr>
<tr>
<td>$P_4$ CDG solution</td>
<td>Lifted $P_6$ solution</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.699E-03 4.95</td>
<td>0.168E-04 6.89</td>
</tr>
<tr>
<td>4</td>
<td>0.221E-04 4.99</td>
<td>0.133E-06 6.98</td>
</tr>
<tr>
<td>5</td>
<td>0.691E-06 5.00</td>
<td>0.106E-08 6.98</td>
</tr>
</tbody>
</table>

References


Department of Mathematics, University of Arkansas at Little Rock, Little Rock, AR 72204, USA.
E-mail address: xxye@ualr.edu

Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA.
E-mail address: szhang@udel.edu